Decision Making under Risk and Uncertainty (Part 1 of 4)

Today's Game Plan

- Risk Pooling
- Making decisions under certainty
- Making decisions under uncertainty: the expected value model
- The St. Petersburg Paradox
- Expected utility model

Risk Pooling

Let $E(L_i)$ = expected loss for insured i and $E(L_T) = \sum_{i=1}^{n} E(L_i)$ = total expected loss of the risk pool. Then

$$E(L_p) = \sum_{i=1}^{n} w_i E(L_i) \qquad (1)$$

= average loss per policy,

where $w_i = E(L_i)/E(L_T)$. Similarly,

$$\sigma_{Lp}^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \rho_{ij} \sigma_i \sigma_j \qquad (2)$$

= average risk per policy, where

$\rho_{ij} = \sigma_i / \sigma_i \sigma_j$ = correlation between losses on policy i and policy j.

Risk Pooling

- Let losses be *identically* distributed; i.e., $E(L_i) = \mu$, $\sigma_i^2 = \sigma^2$, and $w_i = 1/n$ for all insureds, while $\rho_{ij}\sigma_i\sigma_j = \begin{cases} \rho\sigma^2 & \text{if } i \neq j \\ \sigma^2 & \text{if } i = j \end{cases}$. Therefore, (1) and (2) are rewritten

$$E(L_p) = \sum_{i=1}^{n} w_i E(L_i) = (1/n)n\mu = \mu, \text{ and} \tag{1a}$$

$$\sigma_{L_p}^2 = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij}\sigma^2$$

$$= \frac{\sigma^2}{n} + \frac{n-1}{n}\rho\sigma^2. \tag{2a}$$

Risk Pooling

- Since $\sigma_{L_p}^2 = \frac{\sigma^2}{n} + \frac{n-1}{n}\rho\sigma^2$, $\lim_{n\to\infty}\sigma_{L_p}^2 = \rho\sigma^2$; i.e., only *covariance* risk remains.
- Now suppose losses are *iid*. Then $\lim_{n\to\infty}\sigma_{L_p}^2 = 0$. Thus the average loss becomes more predictable as the number of risks pooled becomes large.
  - By pooling many independent risks, insurers can treat uncertain losses as *almost* known.
  - Risk pooling effectively *defrays* risk by exploiting the law of large numbers.

Making decisions under certainty

- Decisions are easier to make under certainty compared with uncertainty!
  - Under certainty, a decision can only lead to one outcome; the best decision involves making a choice that is consistent with one's preferences.
  - e.g., suppose you prefer more money to less and you are faced with two mutually exclusive investment choices that cost $50 each; Investment A pays back $100, whereas Investment B pays back $120.
  - Given your preferences, the logical choice is to invest in B, since B nets $70 whereas A only nets $50.

Making decisions with risk

- In a world where there is risk, decisions are more complicated!
  - With risk, a decision can result in more than one outcome.
  - e.g., suppose Investment A has a 50% chance of paying back $50, and a 50% chance of paying back $150, whereas Investment B has a 50% chance of paying back $0 and a 50% chance of paying back $240. Which investment do you prefer now?
    - Investment A has an *expected* payoff of .5($50) + .5($150) = $100, whereas Investment B has an *expected* payoff of .5($0) + .5($240) = $120. Do you still prefer B over A? Why or why not?

The St. Petersburg Paradox

- Daniel Bernoulli (1738) proposes the following gamble:
  - "Peter tosses a coin and continues to do so until it should land "heads" when it comes to the ground. He agrees to give Paul one ducat if he gets "heads" on the very first throw, two ducats if he gets it on the second, four if on the third, eight if on the fourth, and so on, so that with each additional throw the number of ducats he must pay is doubled. Suppose we seek to determine the value of Paul's expectation."
  - Without any loss of generality, let's change the monetary unit from the ducat to the U. S. dollar.

The St. Petersburg Paradox

- For starters, note that the total probability of an infinite series of fair coin tosses sums to 1 (i.e., $\sum_{i=1}^{\infty} .5^i = 1$), so Professor Bernoulli has proposed a valid probability distribution for this gamble.

- Next, we calculated the expected value of this gamble.
  - If the coin comes up heads on the first toss (when $i = 1$), then the payoff is $\$2^{i-1} = \$2^0 = \$1$, and the game ends. However, if the coin comes up tails on the first toss, it is tossed again.
  - If the coin comes up heads on the second toss (when $i = 2$), the payoff is $\$2^{i-1} = \$2^1 = \$2$, and the game ends. However, if the coin comes up tails on the second toss, it is tossed again, and so forth.

The St. Petersburg Paradox

- The probability that it will take *n* coin tosses in order for heads to come up is $.5^n$, and the payoff after the n^{th} coin toss is $\$2^{n-1}$; thus the expected value of this game is

$$EV = \sum_{i=1}^{\infty} .5^i 2^{i-1} = \sum_{i=1}^{\infty} .5 \Rightarrow \infty .$$

- Bernoulli goes on to note, "My ... cousin discussed this problem in a letter to me asking for my opinion. Although the standard calculation shows that the value of Paul's expectation is infinitely great, it has, he said, to be admitted that any fairly reasonable man would sell his chance, with great pleasure, for twenty ducats (dollars). The accepted method of calculation does, indeed, value Paul's prospects at infinity though no one would be willing to purchase it at a moderately high price."

Solving the Paradox: Expected Utility

- This is how Bernoulli solved his paradox. He assumed an infinite lottery, but introduced the concept of *diminishing marginal utility*; quoting from Bernoulli's paper:

  "The determination of the value of an item must not be based on the price, but rather on the *utility* it yields…. There is no doubt that a gain of one thousand ducats (dollars) is more significant to the pauper than to a rich man though both gain the same amount."

- Bernoulli's utility model involves replacing state-contingent wealth (W_s) with state-contingent utility ($U(W_s) = \ln W_s$). The corresponding infinite series based upon $U(W_s)$ converges in value to a finite sum.

Solving the Paradox: Expected Utility

- What certain payoff, c, would an individual with such a "Bernoulli" utility function regard as equivalent to the St. Petersburg lottery? This c would have to solve the following equation:

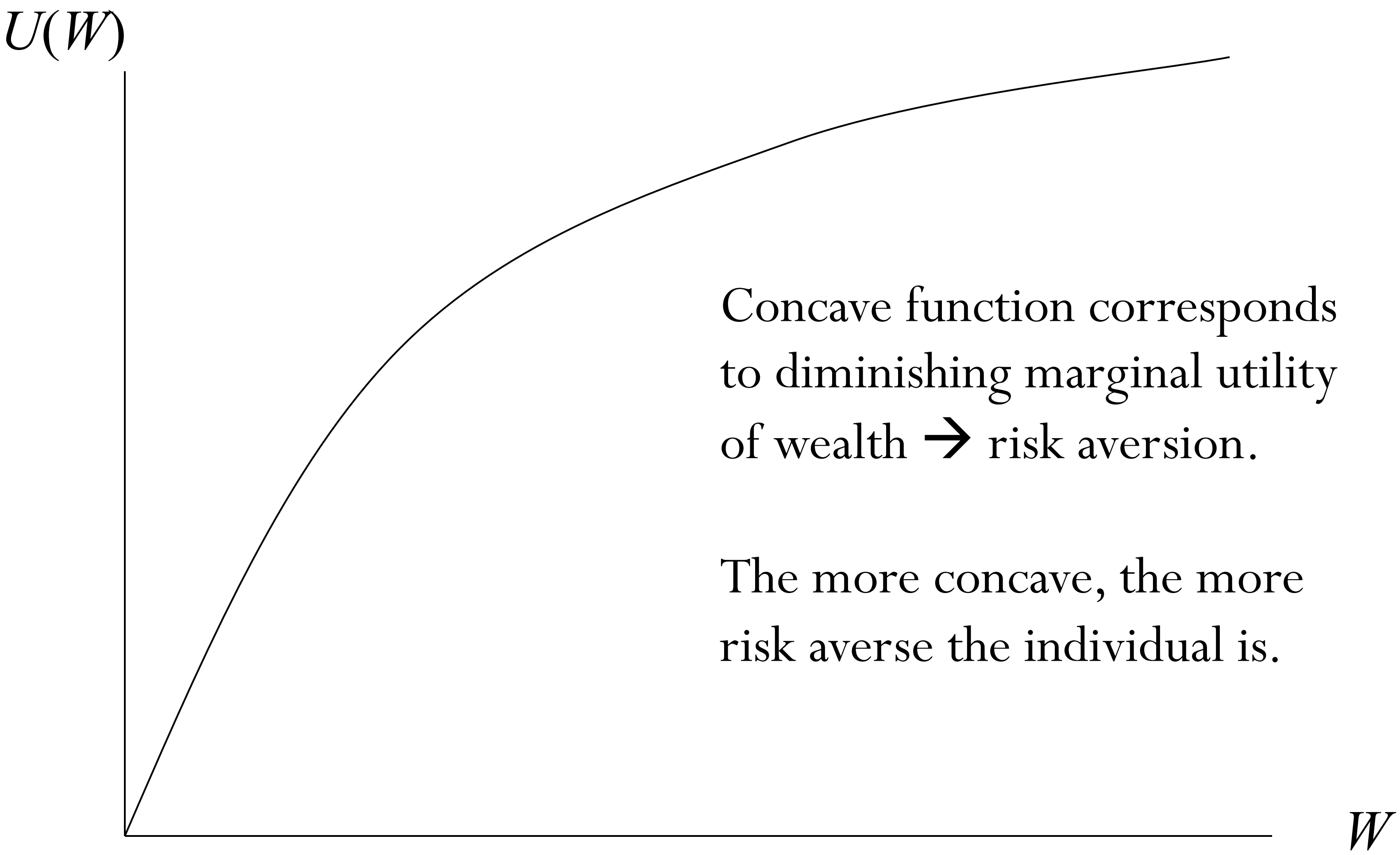
$$\ln(c) = \sum_{i=1}^{\infty} .5^{i} \ln(2^{i-1})$$
$$= \left(\sum_{i=1}^{\infty} (i-1).5^{i} \right) \ln(2)$$
$$= \ln(2)$$

- Thus, $c = 2$ – a rather modest value to place upon Paul's expectation!

Risk Aversion & Diminishing Marginal Utility

- *Risk averse* utility functions have the property of *diminishing marginal utility*.
  - Diminishing marginal utility implies that as a person's wealth increases, there is a decline in the marginal utility that person derives from each additional unit (dollar) of wealth.
  - In other words, the utility value of an additional $1,000 is much higher for someone with a net worth of $1,000 compared with an otherwise identical person with a net worth of $1,000,000.

Graphical Depiction of Risk Aversion



Concave function corresponds to diminishing marginal utility of wealth → risk aversion.

The more concave, the more risk averse the individual is.

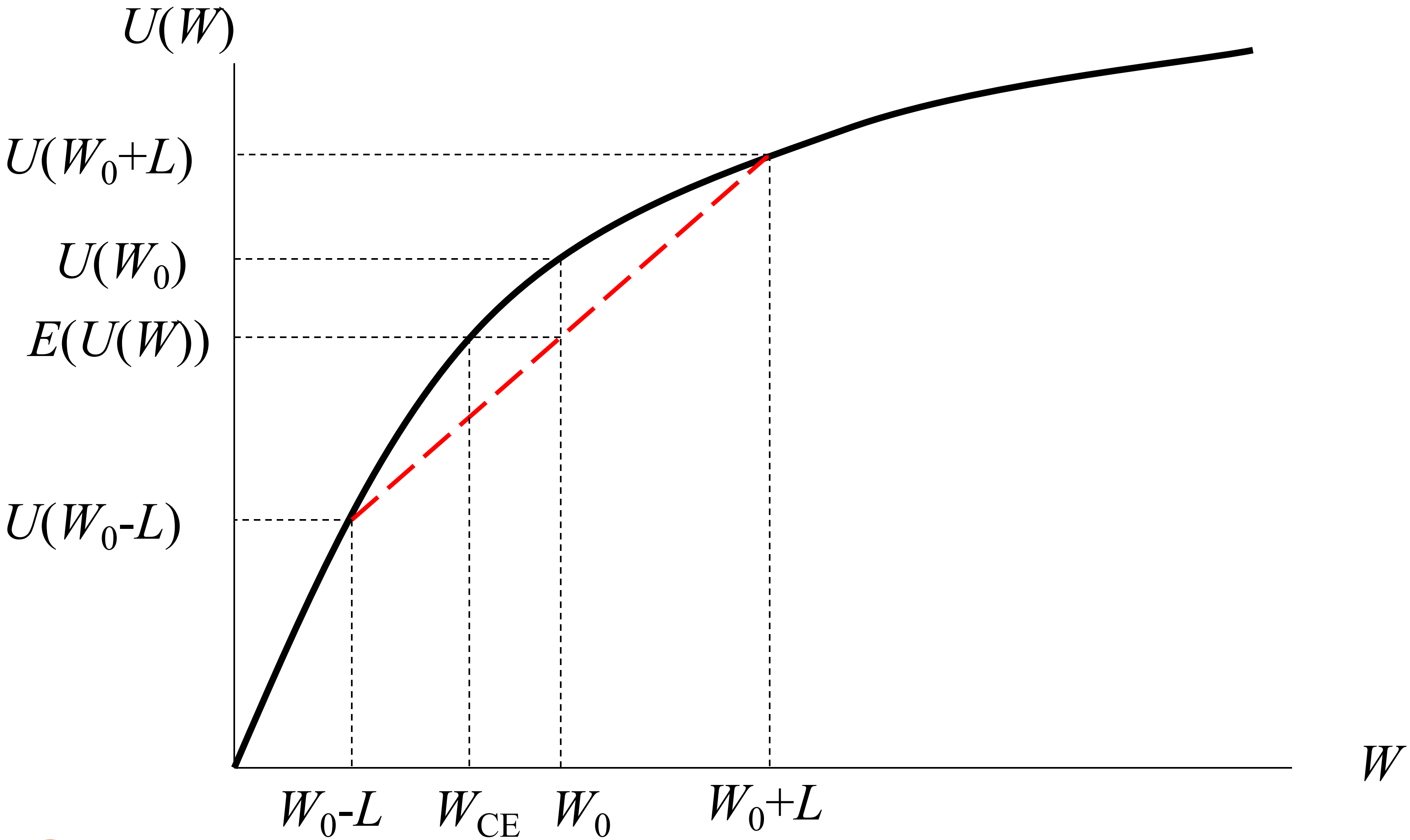
Risk Aversion and Expected Utility

- Most individuals behave (most of the time) *as if* they are *risk averse*.
- Risk averse consumers prefer a certain outcome over an uncertain outcome with the same expected value.
  - If two outcomes with the same expected value are both risky, risk averse consumers prefer the outcome with less risk.
  - If two outcomes are both equally risky, risk averse consumers prefer the one with higher expected value.
- This risk/reward tradeoff is captured by the consumer's *utility function*.

Expected Utility Example

- Suppose your utility function is *logarithmic*; i.e., $U(W) = \ln(W)$ and you start with wealth W_0.
- There is a 50% chance of losing \$$L$ and 50% chance of winning \$$L$ (i.e., this is an "actuarially fair" bet).
- $E(W) = (.5)(W_0 + L) + (.5)(W_0 - L) = W_0$.
- Is expected utility *with* the gamble higher or lower than expected utility without the gamble?
- Is $E(U(W)) = (.5)\ln(W_0 + L) + (.5)\ln(W_0 - L) >, <,$ or $= \ln(W_0)$?
- Note that $U(W)$ is a risk averse utility function!

Certainty Equivalent Wealth



Numerical Example of W_{CE}

- $U(W) = \ln(W)$
- You have initial wealth $W_0 = \$100$, but face a 50% probability of winning \$20, and 50% probability of losing \$20.
- Therefore, $E(U(W)) = .5\ln(80) + .5\ln(120) = 4.585$.
- What amount of certain wealth gives you utility of 4.585?
  - Set $E(U(W)) = U(W_{CE})) \Rightarrow 4.585 = \ln(W_{CE})$.
  - Since $W_{CE} = e^{\ln(W_{CE})}$, this implies that $W_{CE} = e^{4.585} = \$98$.
- Therefore, this person is indifferent between having a 50/50 chance of \$80 or \$120, and having \$98 with certainty!

Risk Loving Behavior

- In the previous set of slides, we showed (among other things) that risk aversion implies that utility is concave; i.e., that $E(U(W)) < U(E(W))$.
- What if utility is *convex*? Suppose $U = W^2$; then $E(U(W)) > U(E(W))$.
- As before, initial wealth W_0 = \$100, there's a 50% probability of winning \$20, and 50% probability of losing \$20.
- Therefore, expected utility = $.5(80^2)+.5(120^2)$ = 10,400.
- What amount of certain wealth gives you utility of 10,400? Set W_{CE}^2= 10,400; therefore, W_{CE} = \$101.98.
- This person would be indifferent between 50/50 chance of \$80 or \$120, and having \$101.98 with certainty!

Risk Neutral Behavior

- What if utility is linear? Suppose $U = W$; then $E(U(W)) = U(E(W))$.
- As before, initial wealth $W_0 = \$100$, there's a 50% probability of winning \$20, and 50% probability of losing \$20.
- Expected utility = $.5(80)+.5(120) = 100$.
- What amount of certain wealth gives you utility of 100? Since $U\,(W_{CE}) = W_{CE} = \$100$, we have our answer!
- This person is indifferent between having a (risky) expected value of \$100 and \$100 with certainty!

Revisiting the Risky Investment Problem!

- Earlier in this lecture, I proposed the following problem:
  - Suppose Investment A has a 50% chance of paying back \$50, and a 50% chance of paying back \$150, whereas Investment B has a 50% chance of paying back \$0 and a 50% chance of paying back \$240. Suppose $U(W) = W^{.5}$ and $W_0 = \$50$. Which investment do you prefer?
  - Which investment do you prefer if $U(W) = W^{.5}$ and $W_0 = \$150$?
  - Which investment do you prefer if $U(W) = \ln(1+W)$ and $W_0 = \$50$? What if $U(W) = \ln(1+W)$ and $W_0 = \$150$?