Teaching the Economics and Convergence of the Binomial and Black-Scholes Option Pricing Formulas

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Abstract

This paper simplifies the economics of option pricing formulas by clarifying how the no-arbitrage principle ensures that a risk-neutral valuation relationship (based on risk-neutral probabilities) exists between an option and its underlying asset. A spreadsheet exercise shows how binomial probabilities and prices numerically converge to Black-Scholes probabilities and prices, and further numerical analysis reveals how the histogram of terminal stock returns in the multi-period binomial tree converges in probability to the normal distribution. Recommendations for teaching option pricing and convergence include the use of a hypothetical case study in the form of a graduating students comparison of competing salary offers.

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1 Introduction

It is often challenging for students of finance to grasp fully the logic of the economics and convergence of the binomial and Black-Scholes option pricing formulas. A rigorous comprehension of these formulas is important not only for investment analysis but also for studying corporate finance topics such as real options, agency theory, risk management, managerial compensation, credit risk, and so on. From a practical perspective, options may also make up an important aspect of future compensation packages for finance graduates. If students understand option pricing theory, they will be better prepared to succeed in their vocational pursuits and personal financial decisions.

In this paper, we provide a pedagogical framework that introduces the basic concepts necessary to understand the economics behind the binomial and Black-Scholes option pricing formulas, and also explains the convergence from the binomial model to the Black-Scholes model. We provide suggestions for walking students through the mathematical portions, and a simple case study example in which students use the binomial and Black-Scholes pricing formulas to evaluate competing salary offers. Finally, we provide a spreadsheet template showing how multi-period binomial model probabilities and prices numerically converge to their Black-Scholes counterparts,[1] and we numerically illustrate how the histogram of the terminal stock return in the multi-period binomial tree numerically converges in probability to the normal density function.

To motivate class discussion, suppose a student receives two competing job offers, and wishes to determine which offer is more financially attractive. Company A has offered a fixed annual salary of \$60,000, whereas Company B's offer is for a fixed annual salary of \$50,000 plus an employee stock option (ESO) grant for 5,000 shares of Company B's stock, expiring in one year with a \$60 per share exercise price. Company B's stock trades for \$50, and in our initial numerical example, the stock price will either rise to \$62.50 or fall to \$40 one year from today. The challenge for the student is to estimate the values of each offer. In subsequent iterations of this example, we replace the binomial outcomes suggested here with outcomes based on the volatility of Company B's stock, and the time to expiration is extended beyond one year. We assume throughout the paper that options are European (i.e., exercise may only occur at expiration), and that the underlying asset

[1]This spreadsheet uses only the standard Excel functions without relying on macros or other challenging coding techniques.

does not pay dividends.[2]

In the next section of this paper, we feature the single-period versions of the delta hedging and replicating portfolio approaches to pricing options, and show how both methods encompass the risk-neutral valuation approach.[3] All three methods rely on the so-called "no-arbitrage" principle, where arbitrage refers to the opportunity to earn riskless profits by taking advantage of price differences between virtually identical investments; i.e., arbitrage represents the financial equivalent of a "free lunch". However, since competition dissipates the opportunity to earn riskless profits, so-called "arbitrage-free" prices for options emerge.

In the section titled "The Multi-Period Model", we extend the risk-neutral valuation model to two or more periods, and show how it generalizes as the Cox, Ross, and Rubinstein (1979) binomial option pricing formula. In the penultimate section of the paper, we illustrate how probabilities and prices under the Cox-Ross-Rubinstein model converge to Black-Scholes option pricing model probabilities and prices, and how terminal stock returns in the multi-period binomial tree numerically converge in probability to the normal density function. We provide concluding remarks in the final section of the paper.

2 The Single-Period Model

Single-period option pricing models based upon delta hedging and replicating portfolio approaches appear in many investment textbooks. We review those models here to introduce our notation, and to provide a complete teaching lesson plan that an instructor can use to illustrate the economic principle of risk-neutral valuation and the convergence from the single-period, binomial options to the continuous-time Black-Scholes option pricing model. The purpose of this section is not to produce new or novel insights about the binomial model; rather, it is to set the stage for use of the

[2]Hull and White (2004) provide technical modifications for the binomial and Black-Scholes option pricing models studied here. They explicitly consider the incremental pricing consequences for employee stock options (ESOs) of vesting periods, the possibility that employees may leave the company during the life of the ESO, and the inability of employees to trade their options. Notwithstanding the practical importance of these issues, a consideration of such unique features of ESOs goes well beyond the scope of this paper. Our primary purpose here is to motivate student interest in studying and understanding the basics of option pricing which are foundational for both the theory and practice of finance.

[3]While leading financial derivatives textbooks by Hull (2015) and McDonald (2013) also emphasize risk-neutral valuation, Hull (pp. 274-280) motivates risk-neutral valuation via the delta hedging approach, whereas McDonald (pp. 293-300) motivates risk-neutral valuation via the replicating portfolio approach. Here, we clarify how the delta hedging and replicating portfolio approaches both represent sufficient conditions for a risk-neutral valuation relationship to exist between an option and its underlying asset.

employee compensation example to illustrate (i) the origin of risk-neutral valuation from the delta hedging and replicating portfolio approaches and (ii) the convergence from the binomial option pricing model to the Black-Scholes option pricing model.

2.1 Delta Hedging Approach

Suppose the student initially applies the delta hedging approach to determine the value of the option component of Company B's compensation offer. The current price per share of Company B's stock is S, and one time-step (δt) from now, the stock will assume one of the following two values: $S_u = uS$or $S_d = dS$, where $u > 1$ and $d < 1$. We assume that $S = \$50$, $u = 1.25$, $d = .8$, $\delta t = 1$ (one year), the exercise price $K = \$60$, and the continuously compounded riskless rate of interest $r = 3\%$. Figure 1 shows the binomial "tree" for the current (known) stock price and also the future (state-contingent) stock prices, and Figure 2 shows the binomial tree for the current (unknown) call option price and also the future (state-contingent) call option prices.

Next, the student forms a "hedge" portfolio comprising a long position in one call option and a short position in Δ shares of stock. This portfolio is called a hedge portfolio because movements in the stock's value hedge, or offset the effect of movements in the call option's value. The current market value of this hedge portfolio is

$$V_H = C - \Delta S = C - \Delta 50. \tag{1}$$

At the up (u) node, the value of the hedge portfolio is equal to $V_H^u = C_u - \Delta S_u = 2.50 - \Delta 62.50$, and at the down ($d$) node, the value of the hedge portfolio is equal to $V_H^d = C_d - \Delta S_d = 0 - \Delta 40$. Suppose we solve for Δ such that the hedge portfolio is riskless; i.e., $V_H^u = V_H^d$. Since $V_H^u = V_H^d$, this implies that $2.50 - \Delta 62.50 = -\Delta 40$ and $\Delta = 0.111$. Substituting $\Delta = 0.111$ back into the expressions for V_H^u and V_H^d, we find that $V_H^u = V_H^d = -\$4.44$. An example of this solution for a whiteboard/presentation slide explanation appears in Figure 3. Thus, the terminal value of a riskless hedge portfolio comprising one call option and a short position in one-ninth of a share of stock is equivalent in value to a *short* position in a "synthetic" riskless bond worth \$4.44 one year from now, and the present value of this short bond position is $V_H = -4.44e^{-.03} = -\$4.31$.

Even though the call option and the stock have completely different cash flow characteristics than a riskless bond, the riskless hedge portfolio comprised of these two securities creates a "synthetic" riskless bond in the sense that its cash flows mimic the riskless bond cash flows. Under no-arbitrage conditions, the price of the synthetic bond must equal the price of the actual bond with the same payoffs. So, for a given stock price, the price of the call option which satisfies this no-arbitrage condition is the arbitrage-free price. Since $V_H = C - (0.111)\,50$, this implies that the arbitrage-free price for the call option is $C = \$1.24$, which implies that the proposed option compensation is worth 5,000 x \$1.24, or \$6,200. Since the value of the Company A's \$60,000 salary-only offer exceeds the value of Company B's salary (\$50,000) plus option (\$6,200) offer, our student will prefer Company A's offer, unless the student is risk-loving or assumes a different probability distribution than the one presented here.

While we would not expect a firm to offer a put option as part of a compensation package to a prospective employee, it is worthwhile to consider how to price an otherwise identical put option with an exercise price of \$60. Since the arbitrage-free price for the call option is \$1.24, we rely upon the put-call parity equation (Stoll (1969)) to determine the arbitrage-free price of an otherwise identical put option.

The put-call parity equation is shown in equation (2):

$$C + Ke^{-r\delta t} = P + S. \tag{2}$$

Thus,

$$P = C + Ke^{-r\delta t} - S = \$1.24 + 60e^{-.03} - \$50 = \$9.47. \tag{3}$$

We can also determine the arbitrage-free price for the put option via the delta hedging approach. Since price movements for a put option and its underlying stock are inversely related, we form a hedge portfolio comprising a long position in one put option and a long position in Δ shares of stock. The current value of this portfolio is

$$V_H = P + \Delta S = P + \Delta 50. \tag{4}$$

At node u, the value of the hedge portfolio is equal to $V_H^u = P_u + \Delta S_u = Max\,(K - 62.50, 0) + \Delta 62.50$ $= 0 + \Delta 62.50$, and at node d, the value of the hedge portfolio is equal to $V_H^d = P_d + \Delta S_d =$

$Max\,(K-40,0)+\Delta 40 = 20+\Delta 40$. Suppose we select Δ such that the hedge portfolio is riskless; i.e., $V_H^u = V_H^d$ implies that $\Delta 62.50 = 20+\Delta 40$; thus $\Delta = 0.889$. Substituting $\Delta = 0.889$ back into the expressions for V_H^u and V_H^d, it follow that $V_H^u = V_H^d = 55.56$. These calculations can be shown on a whiteboard/presentation slide similar to Figure 3. Thus, the terminal value of a riskless hedge portfolio comprising one put option and a long position in eight-tenths of a share of stock is equivalent in value to a *long* position in a synthetic riskless bond worth \$55.56 one year from now. The present value of this long bond position is $V_H = \$55.56e^{-.03} = \$53.91$, which implies that $P =$ \$9.47.

In the next section, we explore an alternative approach to option valuation. Rather than infer the value of an option by pricing a synthetic riskless bond, we infer option value by calculating the values of "synthetic" options created with combinations of the underlying stock and a riskless bond.

2.2 Replicating Portfolio Approach

Another way for the student to determine the value of the call option is to create a replicating portfolio. Under this trading strategy, the student replicates the call option payoffs at nodes u and d by purchasing Δ shares of stock today and financing part of this investment by borrowing money. The current market value of the replicating portfolio must equal the current market value of the option; if the replicating portfolio and the option have different market values, the student can earn positive profits with zero risk and zero net investment by buying the less expensive investment and shorting the more expensive one. Thus, we invoke the no-arbitrage condition to establish that the arbitrage-free price of the call option must equal the value of its replicating portfolio.

To replicate the payoffs of the call option, the student forms a hypothetical portfolio comprising Δ shares of stock and $\$B$ in riskless bonds. The initial cost of forming such a portfolio is $\$\,(\Delta S + B)$. When the option expires, its value depends on whether the stock price goes up or down, as shown in equations (5) and (6):

$$C_u = \Delta uS + e^{r\delta t}B, \text{ and} \tag{5}$$

$$C_d = \Delta dS + e^{r\delta t}B. \tag{6}$$

Note that the first term in equation (5) represents the value of the underlying stock at node u

(uS) multiplied by the number (or fraction) of shares held in the underlying stock. The second term represents the future value of the bond, assuming continuous compounding at the annual rate of r during the δt time interval. Equation (6) provides the corresponding value of the replicating portfolio at node d. The student will determine how many shares to purchase, and how much to borrow by solving equations (5) and (6) for Δ and B:

$$\Delta = \frac{C_u - C_d}{S\,(u-d)} \geq 0, \text{ and} \tag{7}$$

$$B = \frac{uC_d - dC_u}{e^{r\delta t}\,(u-d)} \leq 0. \tag{8}$$

Note that the equalities in equations (7) and (8) only hold when $C_u = C_d = 0$; i.e., only if the call option always expires out of the money. Otherwise, $\Delta > 0$ and B¡0; i.e., node u and d call option payoffs correspond to payoffs at these same nodes on a margined investment in the stock based on the Δ and B values calculated using equations (7) and (8).

Next, let's reconsider these equations in light of our numerical example. From equations (7) and (8), $\Delta = \dfrac{C_u - C_d}{S\,(u-d)} = .111$ and $B = \dfrac{uC_d - dC_u}{e^{r\delta t}\,(u-d)} = \dfrac{1.25\,(0) - .8\,(2.5)}{e^{.03}\,(.45)} = -4.31$. Note that Δ here is the same as the Δ calculated under the delta hedging approach, and the value of B is the same as the value of V_H in the earlier approach. These equations can be worked out on a whiteboard/presentation slide as shown in Figure 4. Thus, the student can replicate the call option by purchasing one-ninth of a share of stock for \$5.55 and borrowing \$4.31. Since the value of the replicating portfolio is $\$\,(\Delta S + B)$ = \$5.55 - 4.31 = \$1.24, this must also be the arbitrage-free value of the call option. Therefore, the decision regarding the choice between Company A's and Company B's compensation offers is exactly the same as the result obtained in the previous section; since Company A's salary-only offer is worth more than Company B's salary plus option compensation package, our student will find Company A's salary offer more financially attractive.

Following similar logic, we can determine the value of the replicating portfolio for the put option. Suppose we form a portfolio comprising Δ shares of stock and $\$B$ in riskless bonds. The initial cost of forming such a portfolio is $\$\,(\Delta S + B)$. At expiration,

$$P_u = \Delta uS + e^{r\delta t}B, \text{ and} \tag{9}$$

$$P_d = \Delta dS + e^{r\delta t}B. \tag{10}$$

Solving equations (9) and (10) for Δ and B, we get:

$$\Delta = \frac{P_u - P_d}{S\,(u-d)} \leq 0, \text{ and} \tag{11}$$

$$B = \frac{uP_d - dP_u}{e^{r\delta t}\,(u-d)} \geq 0. \tag{12}$$

Note that the equalities in equations (11) and (12) only hold when $P_u = P_d = 0$; i.e., only if they put option always expires out of the money. Otherwise, $\Delta < 0$ and $B > 0$; i.e., put option payoffs correspond to payoffs at these same nodes on an investment comprising a short position in the stock, coupled with a long position in a riskless bond based on Δ and B values calculated using equations (11) and (12).

Next, let's reconsider these equations in light of our numerical example. From equations (11) and (12), $\Delta = \frac{P_u - P_d}{S\,(u-d)} = -20/22.50 = -.889$ and $B = \frac{uP_d - dP_u}{e^{r\delta t}\,(u-d)} = \frac{1.25\,(20) - .8\,(0)}{e^{.03}\,(.45)} = \$53.91$. Thus, we can replicate the put option by shorting eight-ninths of a share for \$44.44 and lending \$53.91. Since the value of the replicating portfolio is $\$\,(\Delta S + B) = -\$44.44 + \$53.91 = \$9.47$, this must also be the arbitrage-free price of the put option.

Although the delta hedging and replicating portfolio approaches to option valuation are motivated differently, both approaches yield the same arbitrage-free prices for call and put options. Note that neither the delta hedging approach nor the replicating portfolio approach require the use of probabilities for calculating option prices. This is a somewhat counter-intuitive result, since one would think the value of an option *should* depend upon the probabilities of up and down movements in the value of the underlying stock. This insight is important as we move forward with one more example of a binomial pricing model approach which relies upon risk-neutral, or risk-adjusted probabilities to calculate arbitrage-free option prices. As we show next, this approach is a logical implication of both the delta hedging and risk-neutral valuation approaches.

2.3 Risk-Neutral Valuation Approach

Next, we consider the risk-neutral valuation approach to pricing options. This approach is popular because of its simplicity. However, the most challenging aspect of this approach involves helping students understand where risk-neutral probabilities come from, and what they mean in practice.

Thus far in this section of the paper, we have inferred arbitrage-free prices for call and put options by either creating a synthetic riskless bond (via the delta hedging approach) or by creating synthetic call and put options (via the replicating portfolio approach). Investor risk preferences are not a factor when arbitrage-free prices are formed, because we eliminate risk under both trading strategies. Arbitrage-free prices obtain so long as investors take advantage of opportunities to earn riskless arbitrage profits. Therefore, since the valuation relationship between an option and its underlying asset does not depend upon investor risk preferences, we may price options *as if* investors are *risk-neutral*. This idea is a foundational principle for the risk-neutral valuation approach.

We begin our analysis by showing the relationship which exists between the expected return on the underlying stock (μ), the probability of an up move (p), and the probability of a down move (1-p). Note that

$$E\left(S_{\delta t}\right) = puS + (1-p)\,dS = e^{\mu\delta t}S, \tag{13}$$

where $E\left(S_{\delta t}\right)$ corresponds to the expected value of the stock price at expiration and μ corresponds to the annualized expected return on the stock. Solving equation (13) for p, we find that

$$p = \frac{\left(e^{\mu\delta t} - d\right)}{(u-d)}. \tag{14}$$

We present a whiteboard/presentation slide example of solving for μ from equation (14) in Figure 5.

Suppose that investors are *risk-averse* and that the probability of an up move is $p = 0.60$. Solving equation (14) for μ, we find that $\mu = \frac{\ln\left(pu + (1-p)\,d\right)}{\delta t} = \frac{\ln\left(.6\,(1.25) + (.4)\,.8\right)}{1} = 6.77\%$. Given these probabilities and payoffs, risk-averse investors demand an (annualized) expected rate of return on the risky stock that exceeds the riskless rate of interest by 3.77 percentage points. This additional return over and above the riskless rate of interest corresponds to a risk premium that compensates risk-averse investors for bearing risk.

Now suppose that investors are *risk-neutral*. In a risk-neutral market, the expected return on a risky asset is the same as the expected return on a riskless asset, because risk-neutral investors do not demand a risk premium; i.e., $\mu = r$. Thus, the expected stock price in a risk-neutral market, one time-step from now is:

$$\hat{E}\left(S_{\delta t}\right) = quS + (1-q)\,dS = e^{r\delta t}S, \tag{15}$$

where $\hat{E}\left(S_{\delta t}\right)$ corresponds to the risk-neutral expected stock value, q corresponds to the risk-neutral probability of an up move, and (1-q) corresponds to the risk-neutral probability of a down move. Comparing the right-hand sides of equations (13) and (15), we replace μ with r because $\mu = r$ in a risk-neutral market. Solving equation (15) for q, we find that

$$q = \frac{e^{r\delta t} - d}{(u-d)} = \frac{e^{.03} - .8}{(.45)} = .5121. \tag{16}$$

By using risk-neutral probabilities q and 1-q rather than risk-averse probabilities p and 1-p, this ensures that the risk-neutral expected stock value $\hat{E}\left(S_{\delta t}\right)$ will be less than $E\left(S_{\delta t}\right)$ by an amount that corresponds to the dollar value of the risk premium. Since $E(S_{\delta t}) = Se^{\mu\delta t}$and $\hat{E}(S_{\delta t}) = Se^{r\delta t}$, the dollar value of the risk premium is $E(S_{\delta t}) - \hat{E}(S_{\delta t}) = Se^{(\mu-r)\delta t} = \$50e^{(.0677-.03)1} = \$1.98$. Because q and 1-q are rescaled from p and 1-p in such a way that removes the effect of risk aversion, the initial stock price S can be recovered by discounting $\hat{E}\left(S_{\delta t}\right)$ at the riskless rate of interest; i.e., $S = \hat{E}(S_{\delta t})e^{-r\delta t} = (quS + (1-q)dS)e^{-r\delta t} = (.5121(\$62.50) + .4879(\$40))e^{-.03(??)} = (\$51.52).9704 = \$50$.

Next, we calculate the risk-neutral expected values of the call and put option payoffs at expiration by weighting these payoffs by their corresponding risk-neutral probabilities:

$$\hat{E}\left(C_{\delta t}\right) = qC_u + (1-q)\,C_d, \text{ and} \tag{17}$$

$$\hat{E}\left(P_{\delta t}\right) = qP_u + (1-q)\,P_d, \tag{18}$$

where $\hat{E}\left(\cdot\right)$ corresponds to the risk-neutral expected value operator. Here, $\hat{E}\left(C_{\delta t}\right)$ and $\hat{E}\left(P_{\delta t}\right)$ represent the risk-neutral expected values for the call and put option payoffs at expiration. By discounting $\hat{E}\left(C_{\delta t}\right)$ and $\hat{E}\left(P_{\delta t}\right)$ at the riskless rate of interest, we obtain the current arbitrage-free prices for these (single time-step) European call and put options:[4]

$$C = e^{-r\delta t}\hat{E}\left(C_{\delta t}\right) = e^{-r\delta t}\left[qC_u + (1-q)\,C_d\right] = e^{-.03}\left[.5121\,(5)\right] = \$1.24, \text{ and} \tag{19}$$

$$P = e^{-r\delta t}\hat{E}\left(P_{\delta t}\right) = e^{-r\delta t}\left[qP_u + (1-q)\,P_d\right] = e^{-.03}\left[.4879\,(20)\right] = \$9.47. \tag{20}$$

Since the risk-neutral valuation approach follows as a logical corollary of the delta hedging and replicating portfolio approaches, arbitrage-free prices under risk-neutral valuation must be the same as prices obtained using the delta hedging and replicating portfolio approaches. The decision regarding the choice between the call option or the bonus remains the same as when we created replicating portfolios and synthetic options; since the Company A's salary-only offer is worth more than Company B's salary plus option compensation package, our student will find Company A's compensation offer more financially attractive.

2.4 Risk-neutral Valuation and the Delta Hedging Approach

The student may not understand how three different approaches lead to exactly the same conclusion and wishes to better understand the logical connections that exist between the risk-neutral valuation approach and the delta hedging and replicating portfolio approaches. In the next two sections, we show how delta hedging and portfolio replication imply risk-neutral valuation.

Previously, we formed a hedge portfolio comprising a long position in one call option and a short position in Δ shares of stock. At the beginning of the binomial tree, the hedge portfolio value (as shown by equation (1)) is $V_H = C - \Delta S$. Since $\Delta = \dfrac{C_u - C_d}{S\,(u-d)}$ (see equation (7)),

$$V_H = C - \Delta S = C - \frac{C_u - C_d}{S\,(u-d)}S = C - \frac{C_u - C_d}{(u-d)}. \tag{21}$$

At expiration, the value of the hedge portfolio will be the same, no matter whether the stock moves up or down: i.e., $V_H^u = V_H^d$ implies that $C_u - \dfrac{C_u - C_d}{(u-d)}u = C_d - \dfrac{C_u - C_d}{(u-d)}d$. Thus, the arbitrage-free value of the hedge portfolio, V_H, corresponds to the present value of either V_H^u or V_H^d (let's go with V_H^u): i.e., $V_H = C - \dfrac{C_u - C_d}{(u-d)}$ implies that $C = \dfrac{C_u - C_d}{(u-d)} + e^{-r\delta t}\left[C_u - \dfrac{C_u - C_d}{(u-d)}u\right]$. Solving for the

[4] Note that equations (19) and (20) contain equations (17) and (18) respectively, discounted at the riskless rate of interest.

arbitrage-free price of the call option, we find that

$$\begin{aligned} C &= \frac{C_u - C_d + [(u-d)\,C_u - uC_u + uC_d]\,e^{-r\delta t}}{u-d} \\ &= \frac{C_u - C_d - dC_u e^{-r\delta t} + uC_d e^{-r\delta t}}{u-d} \\ &= e^{-r\delta t}\left[\frac{e^{r\delta t}-d}{u-d}C_u + \frac{u-e^{r\delta t}}{u-d}C_d\right] \end{aligned}$$

$$= e^{-r\delta t}\left[qC_u + (1-q)\,C_d\right]. \tag{22}$$

The risk-neutral valuation relationship shown in equation (22) is identical to the risk-neutral valuation relationship shown in equation (19). Thus, the delta hedging approach implies that a risk-neutral valuation relationship exists between a call option and its underlying stock. By symmetry, the analysis shown here also validates that a risk-neutral valuation relationship exists between a put option and its underlying stock (cf. equation (20)).

2.5 Risk-neutral Valuation and the Replicating Portfolio Approach

Next, we show how the replicating portfolio approach implies risk-neutral valuation. As shown previously, the replicating portfolio was valued $V_{RP} = \Delta S + B$, where $\Delta = \dfrac{C_u - C_d}{S\,(u-d)}$ and $B = \dfrac{uC_d - dC_u}{e^{r\delta t}\,(u-d)}$ (cf. equations (7) and (8)). Thus,

$$\begin{aligned} C &= \frac{C_u - C_d}{S\,(u-d)}S + \frac{uC_d - dC_u}{e^{r\delta t}\,(u-d)} \\ &= \frac{e^{r\delta t}\,(C_u - C_d) + uC_d - dC_u}{e^{r\delta t}\,(u-d)} \\ &= e^{-r\delta t}\frac{C_u\left(e^{r\delta t}-d\right) + C_d\left(u - e^{r\delta t}\right)}{(u-d)}. \end{aligned} \tag{23}$$

Since $q = \dfrac{e^{r\delta t}-d}{u-d}$ and $1-q = \dfrac{u-e^{r\delta t}}{u-d}$, substituting q and $1-q$ into the right-hand side of equation (23) yields:

$$C = e^{-r\delta t}\left[qC_u + (1-q)\,C_d\right]. \tag{24}$$

Thus, the replicating portfolio approach implies that a risk-neutral valuation relationship exists between a call option and its underlying stock. By symmetry, the analysis shown here also validates that a risk-neutral valuation relationship also exists between a put option and its underlying stock (cf. equation (20)).

Now that the logical coherence of the risk-neutral valuation, delta hedging, and replicating portfolio approaches to pricing options in a single-period framework has been shown, our next task involves expanding the risk-neutral valuation model to incorporate multiple periods.

3 The Multi-Period Model

In the previous section of the paper, we assumed that the student's option-based compensation will expire after a single one-year period. In this section, we expand the model to allow for multiple periods prior to expiration. We will expand the risk-neutral valuation model to two or more periods, and then show how it generalizes to the Cox-Ross-Rubinstein binomial option pricing formula.

Suppose that the student now wishes to determine the value of an otherwise identical call option for 5,000 shares of Company B's stock, expiring after *two* one-year periods. Figure 6 shows the binomial tree for the current and future stock prices at the up (u), down (d), up-up (uu), up-down (ud), and down-down (dd) nodes, whereas Figure 7 shows the binomial tree for the current and future call option prices at nodes u, d, uu, ud, and dd. The student will begin at the terminal (uu, ud, and dd) nodes shown in Figure 7, and apply the risk-neutral valuation formula in equation (19) to determine arbitrage-free prices for C_u, C_d, and C. This solution procedure is commonly referred to as "backward induction", since it requires working backwards from the terminal state-contingent values of the call option to the present.

In Figure 7, since the stock only finishes in-the-money at the uu node, $C_{uu} = \$78.13 - \$60 = \$18.13$, whereas $C_{ud} = C_{dd} = \$0$. Thus, the arbitrage-free call option price at node u (applying the node u version of equation (19)), is $C_u = e^{-r\delta t}\left[qC_{uu} + (1-q)\,C_{ud}\right] = e^{-.03}\left[.5121\,(\$18.13)\right] = \$9.01$. Since $C_{ud} = C_{dd} = \$0$, it also follows that $C_d = \$0$. Applying equation (19) once again, the student determines that the current arbitrage-free price of the call option is $C = e^{-r\delta t}\left[qC_u + (1-q)\,C_d\right] = e^{-.03}\left[.5121\,(\$9.01)\right] = \$4.48$.[5] Note that the two-period price is over three times the single-period price of \$1.24. It is well-known that the value of a call option increases as the time to maturity increases. This results from the fact that the underlying asset has more time to increase in value,

[5] Since the two-period call option price is \$4.48, we can determine the price of an otherwise identically configured put option by applying a two-period version of the put-call parity equation given by equation (2); given that $C + Ke^{-r\delta t} = P + S$ for one period, the two-period version of this equation is $C + Ke^{-2r\delta t} = P + S \Rightarrow P = C + Ke^{-2r\delta t} - S \Rightarrow P = \$4.47 + \$60e^{-2(.03)} - \$50 = \$10.98$.

thus increasing the value of the option if it expires in-the-money. Returning to our compensation example, we can see that an otherwise identical call option expiring in two years rather than one year is now worth $22,400, making Company B's offer $12,400 more appealing than Company A's offer.

Although backward induction is required to price the call option via under the delta hedging and replicating portfolio approaches, it is unnecessary under risk-neutral valuation. Since the call option included as part of Company B's compensation package is assumed to be European and may only be exercised at expiration, intermediate node prices for the option (such as C_u and C_d) are not needed to find the current arbitrage-free option price (C), since the value for C depends *solely* on the terminal values of the option. Therefore, the student only needs to undertake the following three steps: 1) calculate the risk-neutral probability for each node at the expiration date, 2) calculate the risk-neutral expected value of the option at expiration, and 3) discount the risk-neutral expected value to present value at the riskless rate of interest for the number of periods to expiration.

The valuation of a multi-period option value (with a few periods) is straightforward for most students. However, understanding that process requires the building blocks shown above (including the delta hedging and replicating portfolio approaches). Once the basic multi-period risk-neutral valuation model is grasped by students, the next step is to introduce them to the Cox, Ross, and Rubinstein (1979) approach to pricing options.

The complexity of analysis grows with each additional time-step. Fortunately, Cox, Ross, and Rubinstein (CRR) greatly simplify the analysis with their recursive multi-period call option pricing formula which appears in equation (25):

$$C = e^{-rT}\left[\sum_{j=0}^{n}\begin{pmatrix} n \\ j \end{pmatrix} q^{j}\left(1-q\right)^{n-j} C_{j}\right]. \tag{25}$$

In equation (25), $\begin{pmatrix} n \\ j \end{pmatrix} = \frac{n!}{j!\,(n-j)!}$ indicates how many j up and $n-j$ down move path sequences exist in an n time-step binomial tree and $T = n\delta t$ corresponds to a fixed expiration date T periods from now. Since $q^{j}(1-q)^{n-j}$ corresponds to the risk-neutral probability of a single j up and $n-j$ down move path sequence, the product $\begin{pmatrix} n \\ j \end{pmatrix} q^j (1-q)^{n-j}$ indicates the risk-neutral probability of the stock price ending up at the $j, n-j$ terminal node.[6] Furthermore, C_j corresponds to the payoff on the call option after n time-steps and j up moves; i.e., $C_j = Max\left[0, u^j d^{n-j} S - K\right]$. The CRR model is considered to be the canonical binomial option pricing model; besides being the best-known and most cited binomial model, the CRR model also provides a simple matching of volatility with the u and d parameters. [7] Furthermore, since $ud = 1.25 \times 0.8 = 1$ in our numerical example, the CRR model implies that $\sigma = \frac{\ln u}{\sqrt{\delta t}} = .2231$.

Here, we recognize that quantitatively challenged students might struggle with understanding the multi-period CRR call option pricing formula in equation (25). Thus, we suggest an optional, brief tutorial for using summation notation in this problem. Suggested whiteboard/presentation slide content appears in Figure 8. Such students might also appreciate a plain-language reading of equation (25), such as, "The value of a call option is the present value of the weighted average of the values of the call option at expiration, where the weightings represent the risk-neutral probabilities of arriving at each terminal node. Thus, today's call option price is the present value of this weighted average, discounted at the riskless rate of interest."

Suppose $n = 1$, in which case there is only one time-step and the length of the time-step is $\delta t = T$. Then equation (25) may be rewritten in the following manner:

$$C = e^{-rT}\left[\sum_{j=0}^{1} \begin{pmatrix} 1 \\ j \end{pmatrix} q^j (1-q)^{1-j} C_j\right] = e^{-rT}\left[(1-q)\,C_0 + qC_1\right] = e^{-rT}\left[(1-q)\,C_d + qC_u\right]. \quad (26)$$

Equation (26) is a special case of equation (25), where $n = 1$. Now suppose that $n = 2$. Then,

$$\begin{aligned} C = e^{-rT}\left[\sum_{j=0}^{2} \begin{pmatrix} 2 \\ j \end{pmatrix} q^j (1-q)^{2-j} C_j\right] &= e^{-rT}\left[(1-q)^2 C_0 + 2q\,(1-q)\,C_1 + q^2 C_2\right] \\ &= e^{-rT}\left[(1-q)^2 C_{dd} + 2q\,(1-q)\,C_{ud} + q^2 C_{uu}\right]. \end{aligned} \quad (27)$$

[6] Trivially, the risk-neutral probabilities associated with the $n+1$ terminal nodes sum to 1; i.e., $\sum_{j=0}^{n} \begin{pmatrix} n \\ j \end{pmatrix} q^j (1-q)^{n-j} = 1.0$.

[7] Specifically, since $u = e^{\sigma\sqrt{\delta t}}$ and $d = e^{-\sigma\sqrt{\delta t}} = \frac{1}{u}$, the variance of stock returns is $\sigma^2 \delta t$ (cf. Hull (2015), pp. 286-287).

Equation (25) can be further simplified by rewriting it in such a way which makes it possible to ignore all cases in which the call option is at- or out-of-the-money. However, we need to know the *minimum* number of "up" moves required during n time-steps in order for this to occur. Since the payoff on the call option after n time-steps and j up moves is $C_j = Max\left(0, u^j d^{n-j} S - K\right)$, we need to determine the minimum (non-negative) integer value for j such that the call option will expire in-the-money; i.e., so that $u^j d^{n-j} S > K$. Let b represent the *non-integer* value for j such that the value of the underlying asset would be equal to K at expiration; i.e., $u^b d^{n-b} S = K$. Solving this equation for b,

$$\begin{aligned} &\ln\left(u^b d^{n-b} S\right) = \ln K \\ &b \ln u + (n-b)\ln d = \ln\left(K/S\right); \\ &b \ln\left(u/d\right) = \ln\left(K/Sd^n\right); \\ &b = \ln\left(K/Sd^n\right)/\ln\left(u/d\right). \end{aligned} \tag{28}$$

Thus, the minimum *integer* value for j such that the call option will expire in-the-money is a, where a is the smallest (non-negative) integer that is greater than a. If $a = 0$, this implies that *all the* call option payoffs at the end of the tree are positive. If $a = n$, then the only node at which a call option pays off is when there have been n consecutive up moves. In theory, a can exceed n; in that case, the call will always expire out of the money and therefore worthless.

Since $u^j d^{n-j} S - K > 0$ for all $j \geq a$, equation (25) can be re-written as follows:

$$C = SB_1 - Ke^{-rT} B_2, \tag{29}$$

where $B_1 = \left[\sum_{j=a}^{n} \begin{pmatrix} n \\ j \end{pmatrix} q^j (1-q)^{n-j} \left(u^j d^{n-j} e^{-rT}\right)\right]$, $B_2 = \left[\sum_{j=a}^{n} \begin{pmatrix} n \\ j \end{pmatrix} q^j (1-q)^{n-j}\right]$, $0 \leq B_1 \leq 1$, and $0 \leq B_2 \leq 1$. Note that B_1 represents the hedge ratio for the binomial option pricing model and B_2 represents the (risk-neutral) binomial probability that the option will expire in-the-money. Furthermore, SB_1 corresponds to today's value of the underlying asset component of the replicating portfolio, whereas $-Ke^{-rT}B_2$ corresponds to today's value of the margin account used to partially finance the underlying asset component of the replicating portfolio.

Equation (29) resembles the Black-Scholes formula for pricing a European call option. The

Black-Scholes formula is given in equation (30):

$$C = SN(d_1) - Ke^{-rT}N(d_2), \tag{30}$$

where $d_1 = \dfrac{\ln(S/K) + (r + .5\sigma^2)T}{\sigma\sqrt{T}}$, $d_2 = d_1 - \sigma\sqrt{T}$, and $N(d_1)$ and $N(d_2)$ correspond to the standard normal distribution function evaluated at d_1 and d_2 respectively. Like B_1 and B_2, $N(d_1)$ and $N(d_2)$ are bounded from below at 0 and from above at 1. Note that in the "limiting" case (where $T = n\delta t$ remains a fixed value as $n \to \infty$ and $\delta t \to 0$), then B_1 converges in value to $N(d_1)$ and B_2 converges in value to $N(d_2)$. Thus, the interpretations offered in the previous paragraph for B_1, B_2, SB_1, and $-Ke^{-rT}B_2$ also apply to $N(d_1)$, $N(d_2)$, $SN(d_1)$, and $-Ke^{-rT}N(d_2)$.

The convergence of the Cox-Ross-Rubinstein binomial option pricing formula in equation (29) and the Black-Scholes option pricing formula in equation (30) can be shown analytically and numerically. For analytic proofs of how probabilities and prices under the CRR binomial model converge to Black-Scholes probabilities and prices, see Cox, Ross, and Rubinstein (1979) and Hsia (1983). Rendleman and Bartter (1979) independently derive a similar binomial model to that of CRR and provide an analytic proof of the convergence of their model to Black-Scholes in an appendix to their paper. Joshi (2011) also considers various binomial models other than CRR and shows that while the CRR $ud = 1$ assumption is analytically convenient, it is unnecessary to get convergence to Black-Scholes. In the next section of the paper, we will *numerically* illustrate the convergence of the CRR model to the Black-Scholes option pricing model, and leave analytic illustration for graduate-level courses.

4 Convergence: Numerical

In a spreadsheet model (available at http://bit.ly/options_econ_converge), we numerically illustrate Black-Scholes and CRR model prices based on our employee stock option example in which $S = \$50$, $K = \$60$, $r = 3\%$, $T = 2$ years, $\sigma = .2231$, and the option is for 5,000 shares of Company B's stock. Applying the Black-Scholes formula provided in equation (30), we find that $d_1 = \dfrac{\ln(S/K) + (r + .5\sigma^2)T}{\sigma\sqrt{T}} = \dfrac{\ln(60/50) + (.03 + .5(.2231^2)\,2}{.2231\sqrt{2}} = -.230, d_2 = d_1 - \sigma\sqrt{T} = -.230 - .2231\sqrt{2} = -.545, N(d_1) = N(-.230) = .409,$ and $N(d_2) = N(-.540) = .293$. Thus, the value of a call option to purchase one share of Company B's stock is \$3.91 (as indicated by the Black-Scholes model), and the value of the option component of Company's compensation offer is \$19,550.

In Table 1, we list CRR model probabilities and prices (based on equation (29)) along with the fixed Black-Scholes model probabilities and price (based on equation (30)) obtained from the spreadsheet model. This table shows that as the number of time-steps increases, the frequency at which the call option expires in-the-money at end-of-tree nodes (as shown by B_2) also varies. The CRR probabilities (as shown in the B_1 and B_2 columns) and CRR model prices swing back and forth as time-steps are added. These swings become less attenuated as the number of time-steps increase, converging toward the Black-Scholes probabilities ($N(d_1) = 0.409$ and $N(d_2) = 0.293$) and \$3.91 price. Figure 9 illustrates the convergence in price and Figure 10 illustrates the convergence in probability. Many of the results obtained from our spreadsheet model (including the "sawtooth" image present in Figure 10) are explained in greater detail by Feng and Kwan (2012).

Similarly, we can show that standardized log returns (with a 0 mean and standard deviation of 1) on the underlying asset also converge to the standard normal distribution. Figure 11 shows histograms and corresponding density functions using the same parameter values as in Table 1 and in Figures 9 and 10, while allowing $n = \{10, 50, 500, \text{and } 5{,}000\}$ and holding $T = n\delta t$ constant. These probability density function charts show convergence from the discrete distribution to the continuous distribution, which follows as a logical consequence of the central limit theorem: as the number of time-steps becomes arbitrarily large, then the discrete distribution converges in probability to the continuous distribution.

5 Conclusion

In this paper, we have provided a simple approach for introducing option pricing models to undergraduate students. We have shown how the delta hedging and replicating portfolio approaches to pricing call and put options imply that risk-neutral valuation relationships exist between option prices and the prices of the underlying assets that they reference. After showing the logical connections between these various approaches in a single-period setting, we show how the risk-neutral approach generalizes to the multi-period case that is captured by the CRR model. Finally, we show

how in the limit (as $n \to \infty$ and $\delta t \to 0$ for a fixed time to expiration), the prices and probabilities which comprise the CRR pricing equation in equation (29) converge to the prices and probabilities which comprise the Black-Scholes pricing equation in equation (30).

To further support instruction of option pricing models, we provide some classroom tools, including a limited prospective employee compensation case study,[8] whiteboard/presentation slide examples that can help instructors explain and show the process to their students and a spreadsheet which shows the convergence between the CRR and Black-Scholes models (available at http://bit.ly/options_econ_converge).

[8]Note that the non-tradability of employee stock options and various vesting rules provide further opportunities to explore modifications to the Black-Scholes model. These are less tractable than the model presented here, but the principles of convergence are the same.

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Table 1. Convergence of Cox-Ross-Rubinstein to Black-Scholes

Time Steps	q	B_1	B_2	CRR Value	$N(d_1)$	$N(d_2)$	Black-Scholes Value
1	0.518	0.669	0.518	\$4.17	0.409	0.293	\$3.91
2	0.512	0.386	0.262	\$4.48	0.409	0.293	\$3.91
3	0.510	0.215	0.132	\$3.29	0.409	0.293	\$3.91
4	0.508	0.452	0.325	\$4.22	0.409	0.293	\$3.91
5	0.507	0.299	0.197	\$3.82	0.409	0.293	\$3.91
10	0.505	0.517	0.390	\$3.83	0.409	0.293	\$3.91
50	0.502	0.360	0.250	\$3.89	0.409	0.293	\$3.91
100	0.502	0.440	0.320	\$3.91	0.409	0.293	\$3.91
200	0.502	0.387	0.273	\$3.91	0.409	0.293	\$3.91
500	0.502	0.408	0.307	\$3.91	0.409	0.293	\$3.91
1000	0.502	0.400	0.285	\$3.91	0.409	0.293	\$3.91
5000	0.502	0.408	0.292	\$3.91	0.409	0.293	\$3.91

Note. – Binomial and Black-Scholes values and risk neutral probabilities of an option with the following parameters: S=50, σ=0.2231, u=1.25, d=0.8, t=2, K=60, r =0.03.

Figures

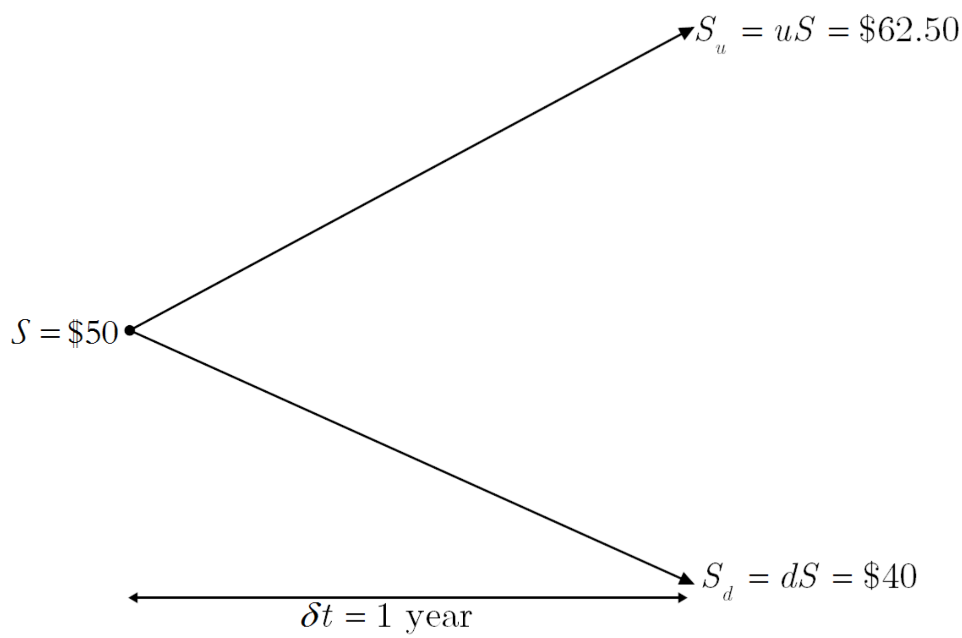

Figure 1. Single-Period Binomial Tree for the Current and Future Stock Prices

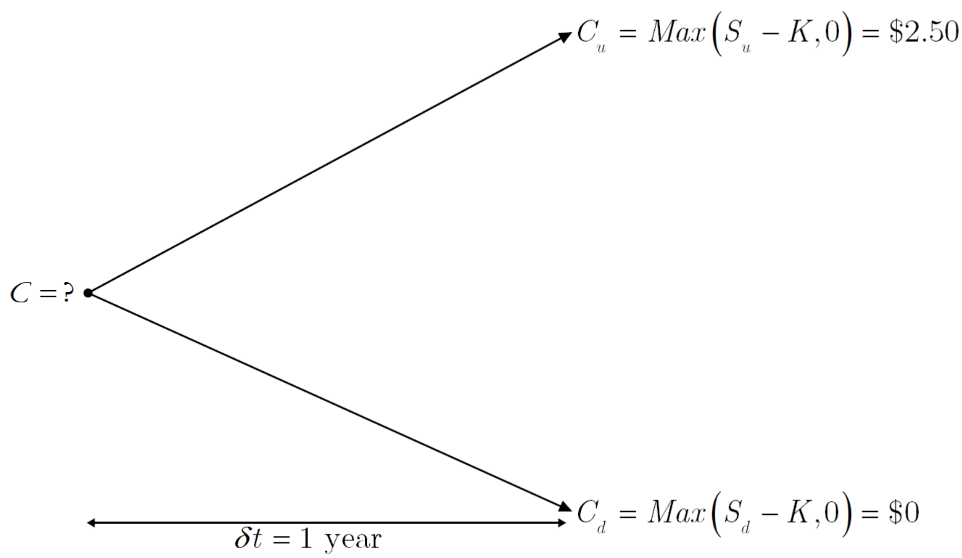

Figure 2. Single-Period Binomial Tree for the Current and Future Call Option Prices

Figures (Continued)

$$C_u - \Delta S_u = C_d - \Delta S_d$$

$$2.50 - \Delta 62.5 = 0 - \Delta 40$$

$$\underline{+\Delta 62.5 \quad + \Delta 62.5}$$

$$2.50 = \Delta 22.5$$

$$\Delta = 0.111 = \frac{1}{9}$$

Then:

$$V_H^u = 2.5 - (0.111)(62.50)$$

$$= 2.5 - 6.9438$$

$$= -4.44 \; and \; ...$$

$$V_H^d = 0 - (0.111)(40)$$

$$= -4.44$$

So:

$$V_H = PV(V_H^u) = PV\left(V_H^d\right) = e^{-0.03}(-4.444) = -4.31$$

Figure 3. Whiteboard Illustration for Finding Hedge Ratio and Present Value of Hedge Portfolio

Figures (Continued)

$$\Delta = \frac{C_u - C_d}{S(u-d)}$$

$$= \frac{2.5 - 0}{50(1.25 - 0.8)}$$

$$= 0.111$$

$$B = \frac{uC_d - dC_u}{e^{r\delta t}(u-d)}$$

$$= \frac{1.25(0) - 0.8(2.5)}{e^{0.03*1*1}(1.25 - 0.8)}$$

$$= \frac{-2}{0.4637}$$

$$= -4.31$$

Figure 4. Whiteboard Illustration for Replicating Portfolio Calculations of Δ and B

$$p = \frac{e^{\mu\delta t} - d}{u - d}$$

$$p(u-d) = e^{\mu\delta t} - d$$

$$pu - pd + d = e^{\mu\delta t}$$

$$pu + (1-p)d = e^{\mu\delta t}$$

$$ln[pu + (1-p)d] = \mu\delta t$$

$$\mu = \frac{ln[pu + (1-p)d]}{dt}$$

Figure 5. Whiteboard Illustration for Deriving Required Return Under Risk (μ)

Figures (Continued)

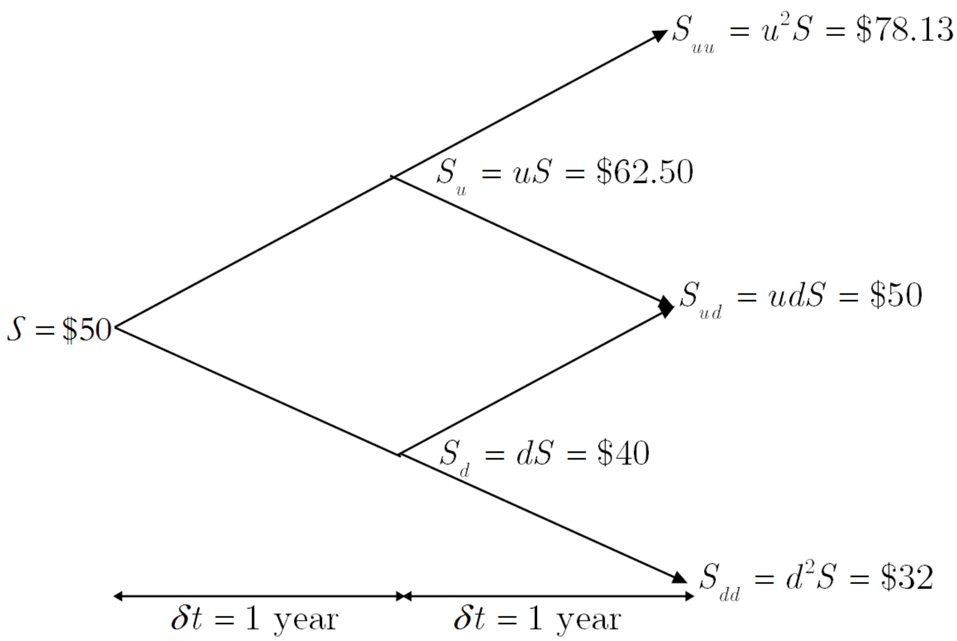

Figure 6. Two-Period Binomial Tree for the Current and Future Stock Prices

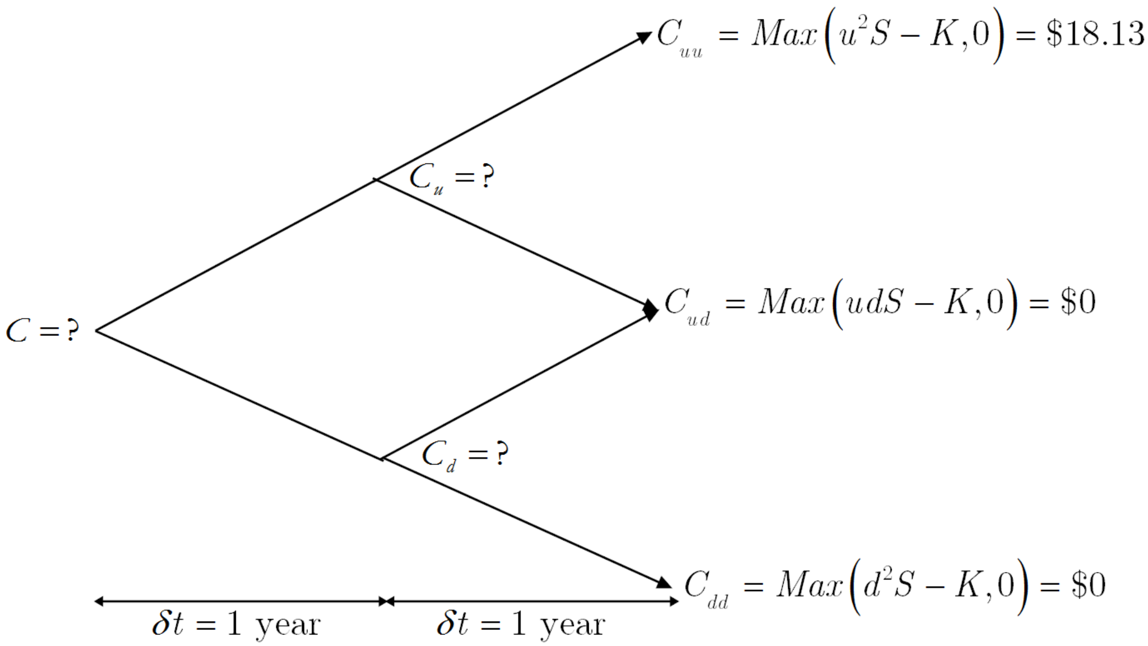

Figure 7. Two-Period Binomial Tree for the Current and Future Call Option Prices

Consider the term in our equation:

$$\sum_{j=0}^{n} \binom{n}{j} q^j (1-q)^{n-j} C_j$$

Where n=the number of time steps, j=the number of up moves to the terminal node, q=risk neutral probability and C_j=the value of the call in terminal node j.

The summation symbol tells us to add the simplified expressions for each j starting with 0 until the number of time steps (2 in our case). So, we will calculate the expression three times (j=0, 1, and 2).

Considering our binomial tree, we know that when j=0 (no up moves, ending in node *dd*), the value of C_j is 0, as the option expires out of the money. The result is similar in our case for j=1 (one up move, ending in node *ud*, which, in a recombining binomial tree, is also node *du*). That leaves j=2 (ending in node *uu*) as the only expression for which we need to simplify the expression.

First, we calculate $\binom{n}{j}$, which is notation for $\frac{n!}{j!(n-j)!} = \frac{2!}{2!(2-2)!} = 1$. Then, we substitute q, n, j, and C_j into the expression:

$$(1)(0.5121^2)(0.4879^{2-2})(18.13) = 4.75$$

Now, we add the three values of this expression: $0 + 0 + 4.75 = 4.75$ and continue solving the equation.

Figure 8. Whiteboard Example: Explanation of Equation (25) Summation Notation

Figures (Continued)

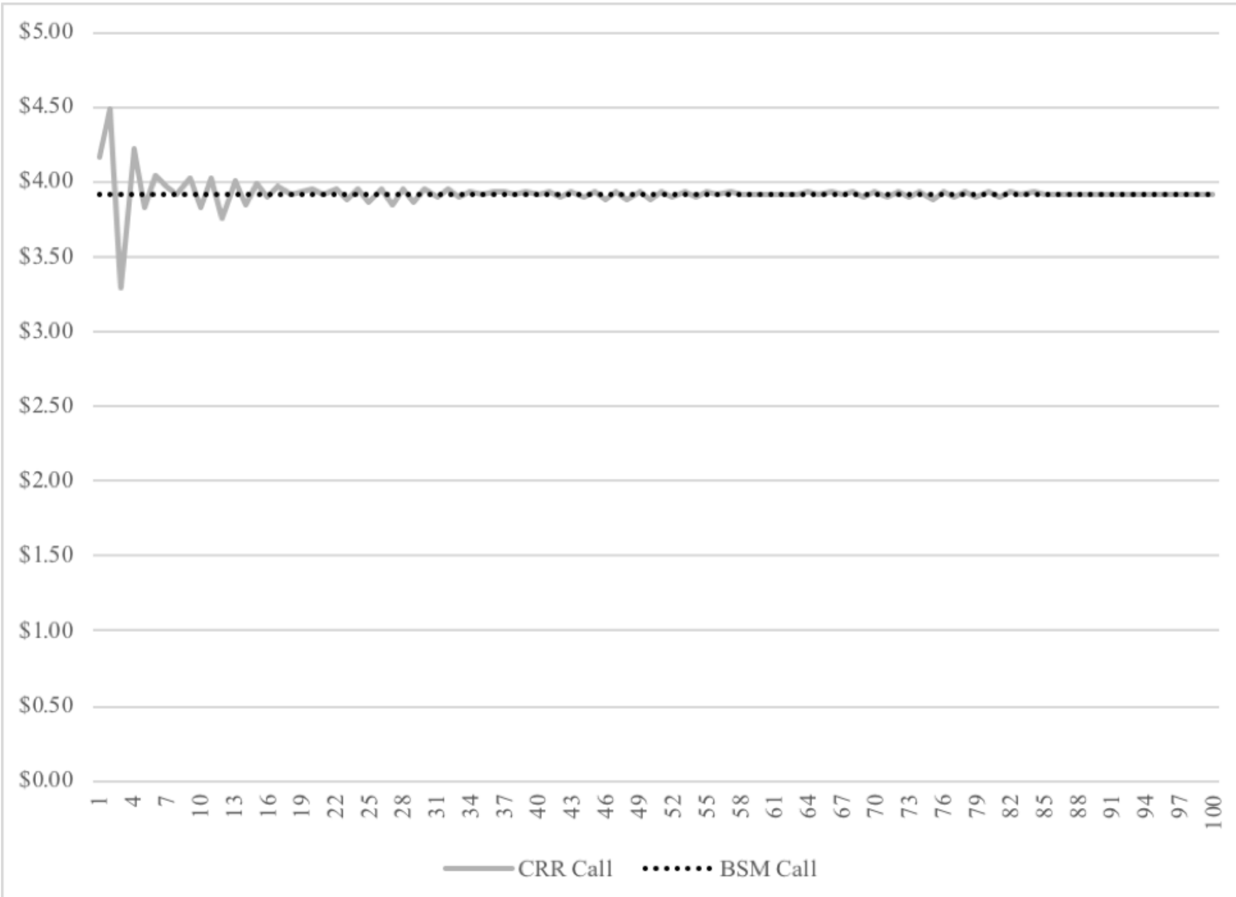

Figure 9. Convergence of Cox-Ross-Rubinstein (CRR) to Black-Scholes Model (BSM) Prices

Note. – Binomial and Black-Scholes values of an option with the following parameters: S=50, σ=0.2231, u=1.25, d=0.8, t=2, K=60, r =0.03. Number of time-steps represented on the x-axis.

Figures (Continued)

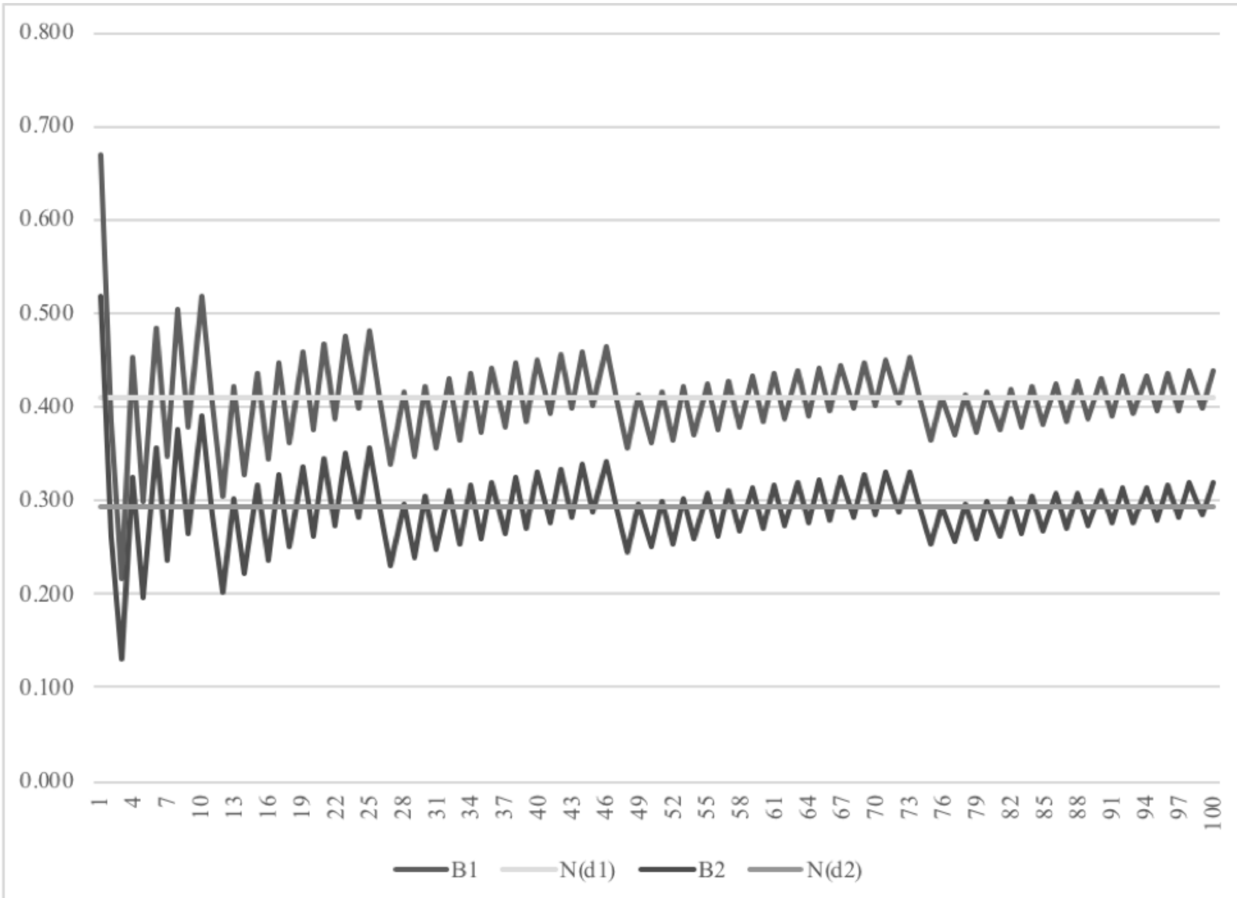

Figure 10. Convergence of Cox-Ross-Rubinstein (CRR) to Black-Scholes Model (BSM) Probabilities

Note. – Binomial and Black-Scholes risk neutral probabilities of an option with the following parameters: S=50, σ=0.2231, u=1.25, d=0.8, t=2, K=60, r =0.03. Number of time-steps represented on the x-axis.

Figures (Continued)

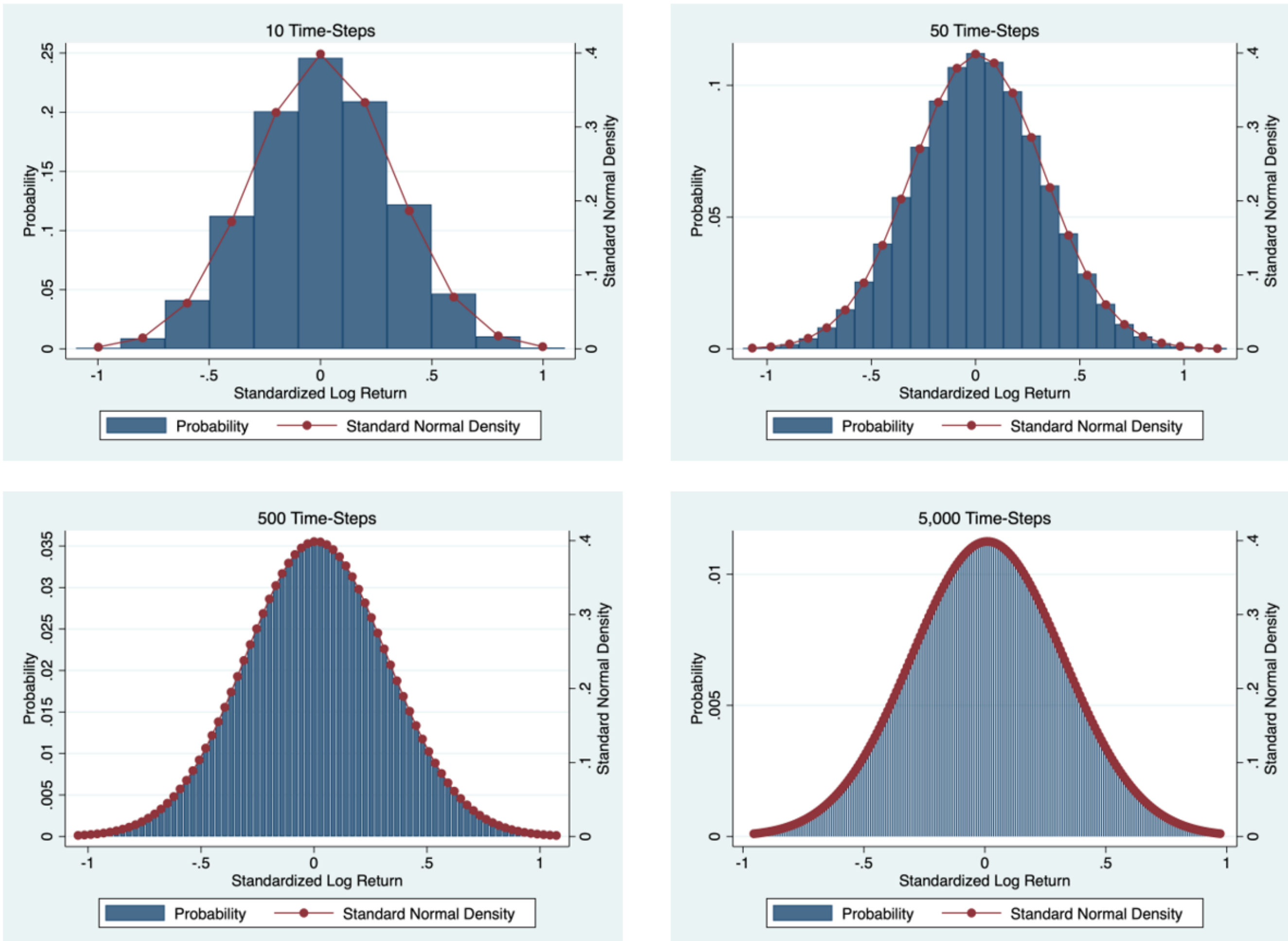

Figure 11. Convergence of Standardized Log Returns under the Binomial Distribution to the Standard Normal Density Function Note. – Parameters used: S=50, σ=0.2231, u=1.25, d=0.8, t=2, K=60, r =0.03. Number of time-steps are 10, 50, 500, and 5,000.