Modeling Risk Preferences Using Taylor Series Expansions of Utility Functions

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1 Introduction

In the undergraduate finance curriculum, the risk-return tradeoff is typically presented as a trade-off between variance and expected value. Commonly referred as the "mean-variance" model, it represents a foundational principle in finance, in that the standard portfolio and capital market theories in finance characterize the risk-return tradeoff in this fashion.

Notwithstanding the importance of the mean-variance model in finance, circumstances arise in which variance by itself is not a particularly appropriate risk measure. For example, lotteries typically offer low probabilities of large payouts. Natural and man-made catastrophes resemble lotteries in that they also feature low probabilities of large payouts. Probability distributions for such risks are often skewed and fat-tailed compared with risks that are normally distributed, and as such are not modeled well by the mean-variance framework.[1]

The purpose of this teaching note is to illustrate the logical connection which exists between the mean-variance and expected utility models, as well as showcase circumstances in which other aspects of risk beyond variance ought to be considered.

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[1]A skewed probability distribution is characterized by a lack of symmetry, or degree to which variable observations pile up on either side of the mean, whereas a probability distribution with fat tails is one in which extreme outcomes are more likely than one would normally expect. In statistics, skewness is measured by calculating the expected value of the cubed deviation of the random variable from its mean, whereas the fat-tailed property is captured by kurtosis, which is measured by calculating the expected value of the quarted deviation of the random variable from its mean. By comparison, a normal distribution is symmetric (i.e., half of all possible variable observations pile up on either side of the mean) and thin-tailed (i.e., extreme outcomes beyond plus and minus three standard deviations are very unlikely).

2 Mean-Variance and Expected Utility Models

The logical connection which exists between the mean-variance and expected utility models is made apparent by considering a second-order Taylor series of an arbitrary risk averse utility function $U(W)$ in which $U(W)$ is approximated for values of W which deviate from $E(W)$:[2]

$$U(W) \cong U(E(W)) + U'(W - E(W)) + (1/2)U''(W - E(W))^2. \tag{1}$$

Next, we find expected utility by calculating the expected value of equation (1):

$$\begin{aligned} E(U(W)) &\cong U(E(W)) + U'E(W - E(W)) + (1/2)U''E(W - E(W))^2 \\ &\cong U(E(W)) + (1/2)U''E(W - E(W))^2 \\ &\cong U(E(W)) + (1/2)U''\sigma_W^2. \end{aligned} \tag{2}$$

Since risk aversion implies that marginal utility is positive ($U' > 0$) and diminishing in wealth ($U'' < 0$), it follows from equation (2) that $E(U(W))$ is positively related to $E(W)$ and negatively related to σ_W^2. In other words, the mean-variance model obtains as a special case of the expected utility model.

3 Mean-Variance-Skewness-Kurtosis & Expected Utility Models

Aspects of risk beyond variance such as skewness and kurtosis are accommodated by considering higher order approximations of expected utility. For example, consider a fourth-order Taylor series approximation of $U(W)$ for values of W which deviate from $E(W)$:

$$\begin{aligned} U(W) \cong\ & U(E(W)) + U'(W - E(W)) + (1/2)U''(W - E(W))^2 \\ & + (1/6)U'''(W - E(W))^3 + (1/24)U''''(W - E(W))^4. \end{aligned} \tag{3}$$

[2] Risk averse utility functions feature positive marginal utility which is diminishing in wealth. There are many such functions which have these properties; e.g., logarithmic (where $U(W) = \ln W$) and power utilities (where $U(W) = W^n$ and $0 < n < 1$), to name a few.

Next, we find expected utility by calculating the expected value of equation (3):

$$\begin{aligned} E(U(W)) &\cong U(E(W)) + U'E(W - E(W)) + (1/2)U''E(W - E(W))^2 + \\ &(1/6)U'''E(W - E(W))^3 + (1/24)U''''E(W - E(W))^4 \\ &\cong U(E(W)) + (1/2)U''\sigma_W^2 + (1/6)U'''Sk_W + (1/24)U''''K_W, \end{aligned} \tag{4}$$

where Sk_W corresponds to skewness and K_W corresponds to kurtosis. Since $U' > 0, U'' < 0, U''' > 0$, and $U'''' < 0$ for risk averse utility functions, it follows that expected utility is positively related to $E(W)$ and Sk_W, and negatively related to σ_W^2 and K_W.

4 Numerical Example

Next, consider the following numerical example. Suppose initial wealth $W_0 = \$0, U(W) = \sqrt{W}$, and state-contingent wealth for two mutually exclusive risks is as follows:

$$W_1 = \begin{cases} 3.5858 \text{ with probability } .45 \\ 6.1571 \text{ with probability } .55 \end{cases} \text{ and } W_2 = \begin{cases} 0 \text{ with probability } .03 \\ 5 \text{ with probability } .94 \\ 10 \text{ with probability } .03 \end{cases}$$

Which risk should one take – risk 1 or risk 2? Since $E(U(W_1)) = .45(\sqrt{3.5858}) + .55(\sqrt{6.1571}) = 2.2169$ and $E(U(W_2)) = .03(\sqrt{0}) + .94(\sqrt{5}) + .03(\sqrt{10}) = 2.1968$, the expected utility model indicates that risk 1 is preferred to risk 2.

Next, we decompose expected utility into its component parts. We begin by calculating means ($E(W_1)$ and $E(W_2)$), variances ($\sigma_{W_1}^2$ and $\sigma_{W_2}^2$), skewness (Sk_{W_1} and Sk_{W_2}), and kurtosis (K_{W_1} and K_{W_2}) for risks 1 and 2:

W_{1s}	p_{1s}	$E(W_1)$ calculation	$\sigma^2_{W_1}$calculation	Sk_{W_1} calculation	K_{W_1} calculation
3.5858	45%	1.61	0.900	-1.273	1.800
6.1571	55%	3.39	0.736	0.852	0.986
		5.00	**1.636**	**-0.421**	**2.786**
W_{2s}	p_{2s}	$E(W_2)$ calculation	$\sigma^2_{W_2}$ calculation	Sk_{W_2} calculation	K_{W_2} calculation
0	3%	0.00	0.750	-3.750	18.750
5	94%	4.70	0.000	0.000	0.000
10	3%	0.30	0.750	3.750	18.750
		5.00	**1.500**	**0.000**	**37.500**

A quick review of this table indicates that 1) risks 1 and 2 share the same mean (5), 2) risk 1's variance is greater than risk 2's variance (1.636 compared with 1.5), 3) risk 1 is negatively skewed ($Sk_{W_1} = -.421$) whereas risk 2 is symmetric ($Sk_{W_2} = 0$), and 4) risk 1 has "thin" tails ($K_{W_1} = 2.786$) whereas risk 2 has "fat" tails ($K_{W_2} = 37.5$).

This numerical example illustrates an important shortcoming of the so-called mean-variance model. Under the mean-variance model, risk is defined as variance and other risk attributes such as skewness and kurtosis are not considered. On the basis of mean and variance only, risk 2 appears to be the better choice. However, there are important differences in skewness and kurtosis between the two risks. Since expected utility is positively related to skewness, risk 1's negative skewness also (in addition to its higher variance) represents an unfavorable attribute. However, risk 1's "thin" tails represent a particularly favorable attribute which outweighs the unfavorable effects of it higher variance and negative skewness. When all four statistical attributes (expected value, variance, skewness, and kurtosis) are taken into consideration, risk 1 is preferred to risk 2.

The following table calculates the determinants of expected utility for risks 1 and 2 based upon the approximation for expected utility given by equation (4); i.e., $E(U(W)) \cong$

$U(E(W)) + (1/2)U''\sigma_W^2 + (1/6)U'''Sk_W + (1/24)U''''K_W$:[3]

$E(U(W_1))$	$U(E(W_1))$	$(1/2)(-.25E(W_1)^{-1.5})\sigma^2_{w_1}$	$(1/6)(.375E(W_1)^{-2.5})Sk_{W_1}$	$(1/24)(-.9375E(W_1)^{-3.5})K_{W_1}$	Sum of Columns 2-5
2.2169	2.2361	-0.0183	-0.0005	-0.0004	2.2169
$E(U(W_2))$	$U(E(W_2))$	$(1/2)(-.25E(W_2)^{-1.5})\sigma^2_{w_2}$	$(1/6)(.375E(W_2)^{-2.5})Sk_{W_2}$	$(1/24)(-.9375E(W_2)^{-3.5})K_{W_2}$	
2.1968	2.2361	-0.0168	0.0000	-0.0052	2.2141

[3]Go to http://fin4335.garven.com/spring2020/riskpref_taylorseries.xlsx in order to download the spreadsheet used to calculate this table.