Technical Note: Stochastic Dominance and Expected Utility

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Here, we formally prove that if $F_A(x)$ stochastically dominates $F_B(x)$, then this implies that $E_A(U(x)) > E_B(U(x))$ for all arbitrarily risk averse utility functions. This proof follows proofs given by Danthine and Donaldson (2002) and Rothschild and Stiglitz (1970).

Without loss of generality, assume that $U(x)$ is defined on the closed interval $[a,\ b]$, and that $U(x)$ is differentiable, with $U'(x) > 0$. Suppose that $E_A(U(x)) > E_B(U(x))$. Since $E_A(U(x)) = \int_a^b U(x)dF_A(x)$ and $E_B(U(x)) = \int_a^b U(x)dF_B(x)$, this implies that $\int_a^b U(x)dF_A(x) - \int_a^b U(x)dF_B(x) > 0$. Next, we simplify the left-hand side of this inequality via integration by parts:[1]

$$\begin{aligned}\int_a^b U(x)dF_A(x) - \int_a^b U(x)dF_B(x) &= U(b)F_A(b) - U(a)F_A(a) - \int_a^b F_A(x)U'(x)dx \\ &\quad - \left\{U(b)F_B(b) - U(a)F_B(a) - \int_a^b F_B(x)U'(x)dx\right\}.\end{aligned}$$

In the expression above, note that $F_A(b)$=$F_B(b)$= 1, whereas $F_A(a)$=$F_B(a)$= 0. Therefore,

$$\begin{aligned}\int_a^b U(x)dF_A(x) - \int_a^b U(x)dF_B(x) &= \int_a^b F_B(x)U'(x)dx - \int_a^b F_A(x)U'(x)dx \\ &= \int_a^b [F_B(x) - F_A(x)]U'(x)dx > 0.\end{aligned}$$

Since marginal utility $U'(x) > 0$, it follows that in order for $\int_a^b [F_B(x) - F_A(x)]U'(x)dx > 0$, then $\int_a^b [F_B(x) - F_A(x)]dx > 0$. However, the condition $\int_a^b [F_B(x) - F_A(x)]dx > 0$ is how

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[1] Integration by parts implies that the solution for

$$\int_a^b udv = uv|_a^b - \int_a^b vdu.$$

we mathematically define second order stochastic dominance. Thus if $F_A(x)$ second order stochastic dominates $F_B(x)$, then this implies that $E_A(U(x)) > E_B(U(x))$. Note also by inspection that if $F_B(x) > F_A(x)$; i.e., if $F_A(x)$ first order stochastic dominates $F_B(x)$, that second order stochastic dominance is also implied, as is the condition $E_A(U(x)) > E_B(U(x))$.

References

Danthine, Jean-Pierre and John B. Donaldson, 2002, *Intermediate Financial Theory*, Prentice Hall.

Rothschild, Michael and Joseph E. Stiglitz, 1970, "Increasing Risk: I. A Definition," *Journal of Economic Theory*, 2: 225-243.