Calculating (Math) Derivatives

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July 19, 2022

In Finance 4335, competency with basic math and stat principles is essential. The Finance 4335 math tutorial is designed to ensure competency with math principles that are needed for understanding how to identify, evaluate, price, and manage risk from personal and corporate perspectives. The 2-part stat tutorial (to be taken up during the second week of class) is designed to ensure competency with stat principles which are used throughout the course.

In this teaching note, I briefly delve into the logical framework required for calculating math derivatives. In math, derivatives measure the rates at which functions' values change with respect to changes in variables. For example, suppose you are interested in determining how the function $y = f(x)$ changes as x changes. The derivative of y with respect to x addresses this question by indicating the value of the function's slope at a given value of x.

1 Example 1 - Slope of a linear function

Suppose we wish to determine the slope of the linear function given by $y = f(x) = a + bx$, as depicted in Figure 1:

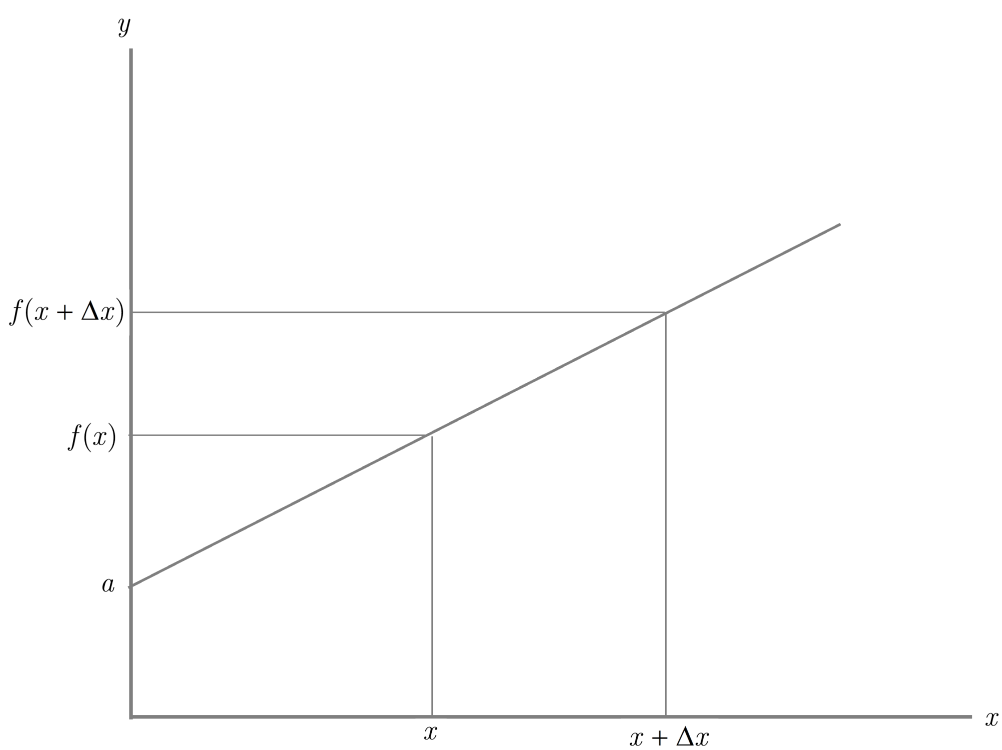

Figure 1: Graph of $y = f(x) = a + bx$.

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The general rule for calculating math derivatives involves applying the "rise over run" principle. Since $y = f(x)$, a Δx change in x (i.e., "run") from x to $x + \Delta x$ produces a corresponding change in y (i.e., "rise") from $f(x)$ to $f(x + \Delta x)$. Thus, the derivative of y with respect to x, i.e., $\frac{dy}{dx}$ is calculated as follows:

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

In the case of the linear function $y = f(x) = a + bx$,

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \left[\frac{(a + bx + b\Delta x) - (a + bx)}{\Delta x} \right] = \frac{b\Delta x}{\Delta x} = b.$$

Thus, the slope of a linear function such as $y = f(x) = a + bx$ is the coefficient b which is multiplied by the variable x. Thus, the slope value b is constant for all values of x.

2 Example 2 - Slopes of non-linear functions

2.1 Slope of a parabola

Now suppose the equation for the function $y = f(x)$ is non-linear; in this case, a parabola; i.e., $y = x^2$. Then

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \left[\frac{(x + \Delta x)^2 - x^2}{\Delta x} \right].$$

We start by expanding the numerator of the ratio which appears on its right-hand side of the equation shown above:

$$\begin{aligned}(x + \Delta x)^2 - x^2 &= x^2 + \Delta x^2 + 2x\Delta x - x^2 \\ &= \Delta x^2 + 2x\Delta x.\end{aligned}$$

Dividing $\Delta x^2 + 2x\Delta x$ by Δx, we obtain

$$\lim_{\Delta x \to 0} \left[\frac{\Delta x^2 + 2x\Delta x}{\Delta x} \right] = \lim_{\Delta x \to 0} (\Delta x + 2x) = 2x.$$

Thus, unlike the result we obtained for our linear equation (where the slope value is constant for all values of x), here we find that the parabola's slope depends on the particular numerical value assumed by the x variable; e.g., if $x = 0$, then $f'(0) = 2(0) = 0$, if $x = 2$, then $f'(2) = 2(2) = 4$, if $x = 4$, then $f'(4) = 2(4) = 8$, and so forth.

2.2 Slope of a power function

Power functions are functions in the form of $y = f(x) = kx^n$, where k is a nonzero coefficient, and n is a real number; note that the parabolic function considered earlier is a power function, where $k = 1$ and $n = 2$.

In the parabola example, we determined that if $y = x^2$, then $\frac{dy}{dx} = 2x^{2-1} = 2x$. In the general case where $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$. This is commonly referred to as the "power rule" of calculus.

3 Partial Differentiation

In sections 1 and 2 of this teaching note, we considered calculus principles for "*univariate*" cases, where y is a function of only one variable. A somewhat more interesting problem involves the so-called *multivariate* case, where the value of a function depends on two or more variables.

Suppose $z = f(x, y)$. Then the *partial* derivative of the function, f, with respect to x (denoted by $\partial f/\partial x$) is:

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \left[\frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \right],$$

and the partial derivative of f with respect to z (denoted by $\partial f/\partial y$) is:

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \left[\frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \right],$$

Here, $\partial f/\partial x$ is the derivative of f with respect to x while holding y constant, whereas $\partial f/\partial y$ is the derivative of f with respect to y while holding x constant.

Numerical Example of Partial Differentiation: Suppose $z = f(x, y) = 2x^2 - 3x^2y + 5y + 1$. Then $\partial f/\partial x$= $4x - 6xy$ and $\partial f/\partial y$=-3x^2 + 5.