CRR Class Problem Set

Tuesday, November 14, 2023

Here are solutions for a very similar problem I had everyone work on during class today, based on implementing the Cox-Ross-Rubinstein formula for a call option, which is:

$$C = e^{-rT}\left[\sum_{j=a}^{n}\frac{n!}{j!\,(n-j)!}q^j(1-q)^{n-j}\left(u^jd^{n-j}S-K\right)\right]$$

Suppose you are interested in determining arbitrage-free prices for a European call option and an (otherwise identical) European put option. The underlying stock does not pay dividends, and its current price is $S = \$50$. For both options, the exercise price $K = \$75$, $u = e^{\sigma\sqrt{\delta t}}$, and $d = e^{-\sigma\sqrt{\delta t}}$. Furthermore, the riskless rate of interest $r = 3\%$ per year, the underlying stock's volatility $\sigma = 25\%$ per year, and both options expire 1 year from today; i.e., the time to expiration is $T = n\delta t = 1$.

A. (30 points) What is the arbitrage-free price for this call option?

SOLUTION: The first step in solving this problem involves finding the value for "a", corresponding to the minimum number of up moves required for the call option to expire in the money. Specifically, $a =$ the smallest integer value $> \ln(K/Sd^n)/\ln(u/d)$. Since $\sigma = 25\%$ and $\delta t = 1/8$, $u = e^{\sigma\sqrt{\delta t}} = e^{.25\sqrt{1/8}} = 1.0924$ and $d = 1/u = .9154$, then $\ln(75/50(.9154^8))/\ln(1.0924/.9154) = 6.29$, so $a = 7$.

Next, we calculate $q = \dfrac{e^{r\delta t}-d}{u-d} = \dfrac{e^{.03/8}-.9154}{1.0924-.9154} = .4991$. The risk neutral probability for a single path to a terminal node where there are 7 up moves and 1 down move is $q^j(1-q) = q^7(1-q) = .4991^7(.5009) = 0.3866\%$. Since the number of path sequences to such a node is $\dfrac{n!}{j!\,(n-j)!} = \dfrac{8!}{7!1!} = 8$, it follows that the risk neutral probability of 7 up moves and 1 down move is $8(0.3866\%) = 3.093\%$. The payoff on the call option at such a node is $u^jd^{n-j}S - K = 1.0924^7(.9154)\$50 - \$75 = \$84.97 - \$75 = \$9.97$. The only other terminal node in which the call is in the money is after 8 consecutive up moves. The risk neutral probability of 8 consecutive up moves is $q^8 = .4991^8 = .3853\%$, and the payoff on the call option after 8 consecutive up moves is $u^8S - K = 1.0924^8(\$50) - \$75 = \$101.41 - \$75 = \$26.41$. Therefore, the arbitrage-free value of this call option is:

$$C = e^{-.03}\left[.03093(9.97) + .003853(\$26.41)\right] = \$.40.$$

B. (20 points) What is the arbitrage-free price for the put option?

SOLUTION: Applying the put-call parity equation, we find that $c + Ke^{-rn\delta t} = p + S \Rightarrow p = \$.40 + \$75e^{-.03} - 75 = \$23.18$.