Decision Making under Risk and Uncertainty (Part 3 of 4)

"Take calculated risks. That is quite different from being rash."

General George S. Patton

Today's Agenda

- Broader Definitions of Risk and Risk Preference

What We Learned Last Time

- Expected utility theory enables us to model how investors make tradeoffs between risk and reward.
- Arrow-Pratt measures of risk aversion
  - Absolute risk aversion ($R_A(W) = -U''(W)/U'(W)$).
    - By measuring the utility function's curvature, $R_A(W)$ indicates the investor's *degree of risk aversion* at a given level of wealth.
  - Relative risk aversion ($R_R(W) = WR_A(W)$).
    - $R_R(W)$ measures how the proportion of wealth that the investor is willing to risk changes as wealth changes.

What We Learned Last Time

- Empirically, investors typically exhibit decreasing absolute risk aversion (*DARA*) and constant relative risk aversion (*CRRA*).
- Risk premiums depend upon an objective factor (i.e., how "risky" the risk is, as measured by variance) and a subjective factor (how risk averse the risk taker is, as measured by the Arrow-Pratt absolute risk aversion coefficient).

Expected Utility Origin of the Mean-Variance Model

- Recall from the last lecture that a Taylor series expansion of $U(W)$ centered at $W=E(W)$ gives rise to the following equation:

$$U(W) \cong U(E(W)) + U'(W - E(W)) + .5U''(W - E(W))^2.$$

- Taking expectations of both sides of this equation, it follows that $E(U(W)) \cong U(E(W)) + .5U''\sigma_W^2$.
- Since $E(U(W))$ is positively related to expected value and inversely related to variance, then $E(U(W))$ can be maximized if one maximizes $E(W)$ (minimizes variance) for a given level of variance ($E(W)$); this is logical foundation for the so-called mean-variance (MV) model in finance.

Contradiction between *EU* & *MV* Models

Consider two risky prospects, X_1 and X_2, with payoffs given by:

$$X_1 = \begin{cases} 1 \text{ with probability } 1/2 \\ 9 \text{ with probability } 1/2 \end{cases} \text{ and}$$

$$X_2 = \begin{cases} 4 \text{ with probability } 99/100 \\ 81 \text{ with probability } 1/100 \end{cases}$$

Assume that your initial wealth (W_0) is \$0, and your utility $U(W)$ $= \sqrt{W}$.

Contradiction between *EU* & Mean-Variance Models

- Which prospect is preferred according to the mean-variance (*MV*) model?

<u>SOLUTION:</u> $E(W_1) = .5(1) + .5(9) = 5$ and

$$\sigma_{W_1} = \sqrt{.5(1-5)^2 + .5(9-5)^2} = 4, \text{ whereas}$$

$$E(W_2) = .99(4) + .01(81) = 4.76 \text{ and}$$

$$\sigma_{W_2} = \sqrt{.99(4-4.77)^2 + .01(81-4.77)^2} = 7.66.$$

According to the mean-variance model, risk 1 is preferred to risk 2 because and $E\left(W_1\right) > E\left(W_2\right)$ and $\sigma_{W_1} < \sigma_{W_2}$.

Contradiction between *EU* & Mean-Variance Models

- However, note that $E(U(W_1)) = .5(1) + .5(3) = 2,$ and $E(U(W_2)) = .99(2) + .01(9) = 2.07; \therefore E(U(W_1)) < E(U(W_2)).$
- Why the apparent conflict between *MV* and *EU*?
  - Note that risk 2 is highly positively skewed (i.e., it provides a *small* chance of a really *large* payoff), whereas risk 1 is symmetric about its mean.
  - Apparently risk 2's positive skewness effect more than offsets its negative expected value and variance effects.
  - Here, the *MV* model does not provide an appropriate framework for risk analysis since it ignores a favorable risk attribute which is implicitly captured by the *EU* model, which is positive skewness.

Broader Definitions of Risk and Risk Preference

- Most of your previous analysis of reward and risk (prior to this course) has focused upon the tradeoff between mean and variance.
  - Other things equal, a higher mean return is preferred to a lower mean return, and a lower variance is preferred to a higher variance.
- *EU* theory encompasses mean-variance analysis as a special case while further challenging us to think more carefully about the nature of risk; while variance is an important risk attribute, so are other characteristics of probability distributions such as skewness and kurtosis.

Broader Definitions of Risk and Risk Preference

- Expected value, variance, skewness, and kurtosis provide us with the following information:
  - Expected value represents the *mean*, or central value about which variable observations scatter;
  - Variance/Standard Deviation indicate how far most of the variable observations scatter about the mean;
  - Skewness indicates the lack of symmetry, or the degree to which variable observations pile up on either side of the mean; and
  - Kurtosis indicates how far variable observations scatter from the mean.

Broader Definitions of Risk and Risk Preference

- Recall that for any function $f(x)$, we can characterize the approximate value of that function evaluated at the point $x + \delta x$ using a Taylor series expansion:

$$f(x+\delta x) \approx f(x) + \sum_{i=1}^{n} \frac{1}{i!} \delta x^{i} \frac{d^{i} f(x)}{dx^{i}} + R^{n+1},$$

where R^{n+1} is a "remainder term".

- Using a Taylor series, we can characterize the value of $U(W)$ around the expected value of wealth, $E(W)$:

$$U(W) = U(E(W)) + U'(W - E(W)) + (1/2)U''(W - E(W))^{2}$$
$$+ (1/6)U'''(W - E(W))^{3} + (1/24)U''''(W - E(W))^{4} + R^{5}.$$

Broader Definitions of Risk and Risk Preference

- Since we are interested in the expected utility of wealth, we compute this by applying the expected value operator to both sides of the previous equation (we also assume that R^5 is negligible):

$$E(U(W)) = U(E(W)) + U'E(W - E(W)) + (1/2)U''E(W - E(W))^2$$
$$+ (1/6)U'''E(W - E(W))^3 + (1/24)U''''E(W - E(W))^4$$
$$= U(E(W)) + (1/2)U''\sigma_W^2 + (1/6)U'''Sk_W + (1/24)U''''K_W.$$

- Thus, expected utility is a function of expected wealth ($E(W)$), variance (σ_W^2), skewness (Sk_W), and kurtosis (K_W).

Broader Definitions of Risk and Risk Preference

- An interesting question concerns how expected utility varies with respect to changes in expected wealth ($E(W)$), variance, skewness, and kurtosis.

$$\frac{\partial E(U(W))}{\partial E(W)} = U'(E(W)) > 0 \text{ (expected wealth preference);}$$

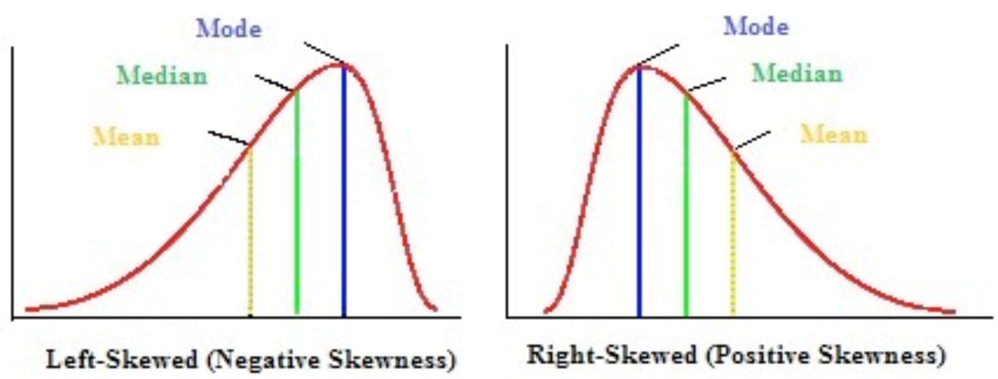
$$\frac{\partial E(U(W))}{\partial \sigma_W^2} = \frac{1}{2}U'' < 0 \text{ (variance preference);}$$

$$\frac{\partial E(U(W))}{\partial Sk_W} = \frac{1}{6}U''' > 0 \text{ (skewness preference); and}$$

$$\frac{\partial E(U(W))}{\partial K_W} = \frac{1}{24}U'''' < 0 \text{ (kurtosis preference).}$$

Broader Definitions of Risk and Risk Preference

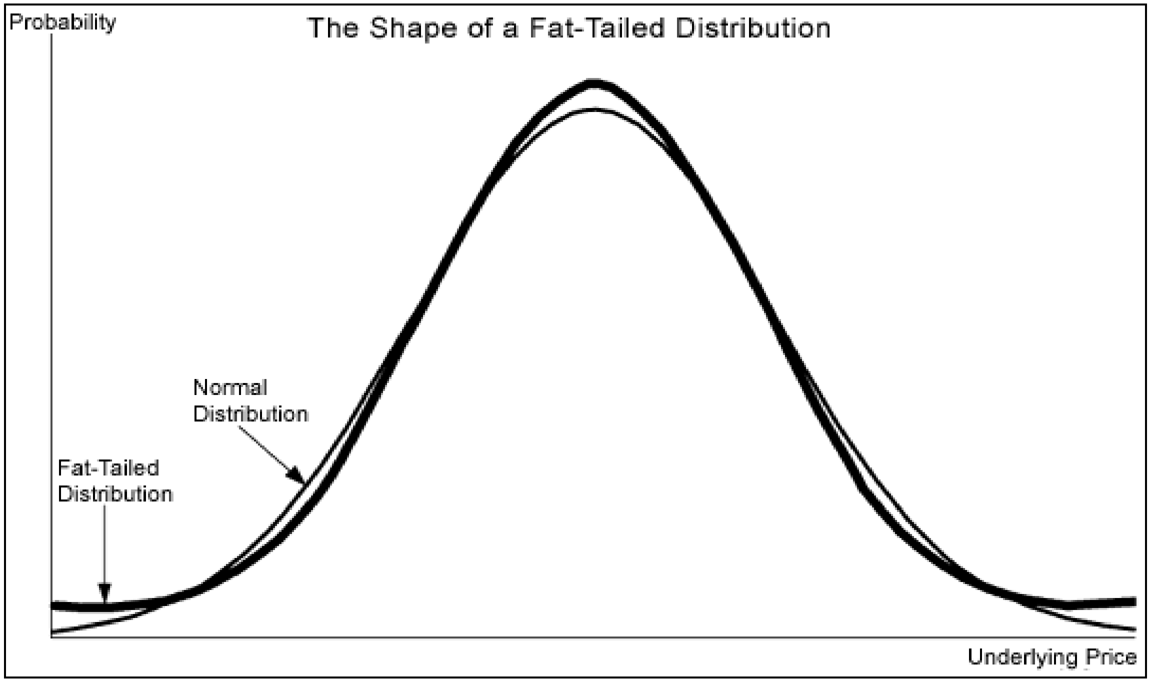
- Since $Sk_W = E(W - E(W))^3$, Sk_W can be negative, zero, or positive. Here are pictures of negatively skewed and positively skewed distributions:



- *EU* theory implies if these two distributions have the same expected value, variance, and kurtosis, then the positively skewed distribution is preferred by an arbitrarily risk averse investor to the negatively skewed distribution.

Broader Definitions of Risk and Risk Preference

- According to *EU* theory, if two risks have the same expected value, variance, and skewness, then the more "thin-tailed" (lower kurtosis) risk is preferred by an arbitrarily risk averse investor to the fat-tailed (higher kurtosis) risk; e.g., consider the following picture:



Numerical Example

Suppose initial wealth $W_0 = \$0, U(W) = \sqrt{W}$, and state-contingent wealth for two mutually exclusive risks is as follows:

$$W_1 = \begin{cases} 3.5858 \text{ with probability } .45 \\ 6.1571 \text{ with probability } .55 \end{cases} \text{ and } W_2 = \begin{cases} 0 \text{ with probability } .03 \\ 5 \text{ with probability } .94 \\ 10 \text{ with probability } .03 \end{cases}$$

Which risk should one take – risk 1 or risk 2? Since $E(U(W_1)) = .45(\sqrt{3.5858}) + .55(\sqrt{6.1571}) = 2.2169$ and $E(U(W_2)) = .03(\sqrt{0}) + .94(\sqrt{5}) + .03(\sqrt{10}) = 2.1968$, the expected utility model indicates that risk 1 is preferred to risk 2.

Numerical Example

W_{1s}	p_{1s}	$E(W_1)$ calculation	$\sigma^2_{W_1}$ calculation	Sk_{W_1} calculation	K_{W_1} calculation
3.5858	45%	1.61	0.900	-1.273	1.800
6.1571	55%	3.39	0.736	0.852	0.986
		5.00	**1.636**	**-0.421**	**2.786**
W_{2s}	p_{2s}	$E(W_2)$ calculation	$\sigma^2_{W_2}$ calculation	Sk_{W_2} calculation	K_{W_2} calculation
0	3%	0.00	0.750	-3.750	18.750
5	94%	4.70	0.000	0.000	0.000
10	3%	0.30	0.750	3.750	18.750
		5.00	**1.500**	**0.000**	**37.500**

Numerical Example

$E(U(W_1))$	$U(E(W_1))$	$(1/2)(-.25E(W_1)^{-1.5})\sigma^2_{w_1}$	$(1/6)(.375E(W_1)^{-2.5})Sk_{W_1}$	$(1/24)(-.9375E(W_1)^{-3.5})K_{W_1}$	Sum of Columns 2-5
2.2169	2.2361	-0.0183	-0.0005	-0.0004	2.2169
$E(U(W_2))$	$U(E(W_2))$	$(1/2)(-.25E(W_2)^{-1.5})\sigma^2_{w_2}$	$(1/6)(.375E(W_2)^{-2.5})Sk_{W_2}$	$(1/24)(-.9375E(W_2)^{-3.5})K_{W_2}$	
2.1968	2.2361	-0.0168	0.0000	-0.0052	2.2141