Decision Making under Risk and Uncertainty (Part 2 of 4)

What we learned last time

- The decision maker's attitude toward risk is described by the relation between $U(E(W))$ and $E(U(W))$.
  - If $E(U(W)) > U(E(W))$, then the decision maker is *risk loving*, since she prefers risk over certainty.
  - If $U(E(W)) > E(U(W))$, then the decision maker is *risk averse*, since she prefers certainty over risk.
  - If $U(E(W)) = E(U(W))$, then the decision maker is *risk neutral*, since she is indifferent between risk and certainty.

What we learned last time

- Certainty-equivalent of wealth – after calculating expected utility, set $E(U(W)) = U(W_{CE})$, and solve for W_{CE}.
  - Example from last time: $U(W) = \ln W$ and $E(U(W)) = 4.585$; therefore, $U(W_{CE}) = \ln W_{CE}$, which implies that $W_{CE} = e^{4.585} = \$98$.
- Risk Premium (λ) equals the difference between expected wealth and the certainty-equivalent of wealth; i.e., $\lambda = E(W) - W_{CE} = \$100 - \$98 = \$2$.
- λ is inversely related to initial wealth; e.g., in this problem, if we start with initial wealth of \$200 rather than \$100, then $E(U(W)) = 5.2933$, $W_{CE} = \$199$, and $\lambda = \$1$.

Arrow-Pratt Risk Aversion Measures

- Arrow-Pratt risk aversion measures provide computational shortcuts for calculating risk premiums which are implied by various risk averse utility functions.
- Arrow-Pratt risk aversion measures indicate the *degree* to which people are averse, and the manner in which one's degree of risk aversion is related to changes in wealth.

Arrow-Pratt Risk Aversion Measures

- We begin by setting $E(U(W)) = U(W_{CE})$; since $W_{CE} = E(W) - \lambda(E(W))$, the following equation obtains:

$$E(U(W)) = U(W_{CE}) = U[E(W) - \lambda(E(W))].$$

- Next, we linearly approximate $U[E(W) - \lambda(E(W))]$ with a first-order (linear) Taylor series expansion; i.e.,

$$U[E(W) - \lambda(E(W))] \cong U(E(W)) - \lambda(E(W))U'.$$

Arrow-Pratt Risk Aversion Measures

- We now turn our attention to $E(U(W))$; specifically, consider a second-order (quadratic) Taylor series expansion of $U(W)$ centered at $W=E(W)$:

$$U(W) \cong U(E(W)) + U'(W - E(W)) + .5U''(W - E(W))^2.$$

- Taking expectations of the above equation, we now have an approximation for $E(U(W))$:

$$\begin{aligned} E(U(W)) &\cong U(E(W)) + U'E(W - E(W)) + .5U''E(W - E(W))^2 \\ &= U(E(W)) + .5U''\sigma_W^2. \end{aligned}$$

Arrow-Pratt Risk Aversion Measures

- Since $U(W_{CE}) \cong U(E(W)) - \lambda(E(W))U'$ and $E(U(W)) \cong U(E(W)) + .5U''\sigma_W^2$, setting $E(U(W)) = U(W_{CE})$ yields the following expression for the risk premium $\lambda(E(W))$:

$$U(E(W)) + .5U''\sigma_W^2 = U(E(W)) - \lambda(E(W))U' \Rightarrow$$

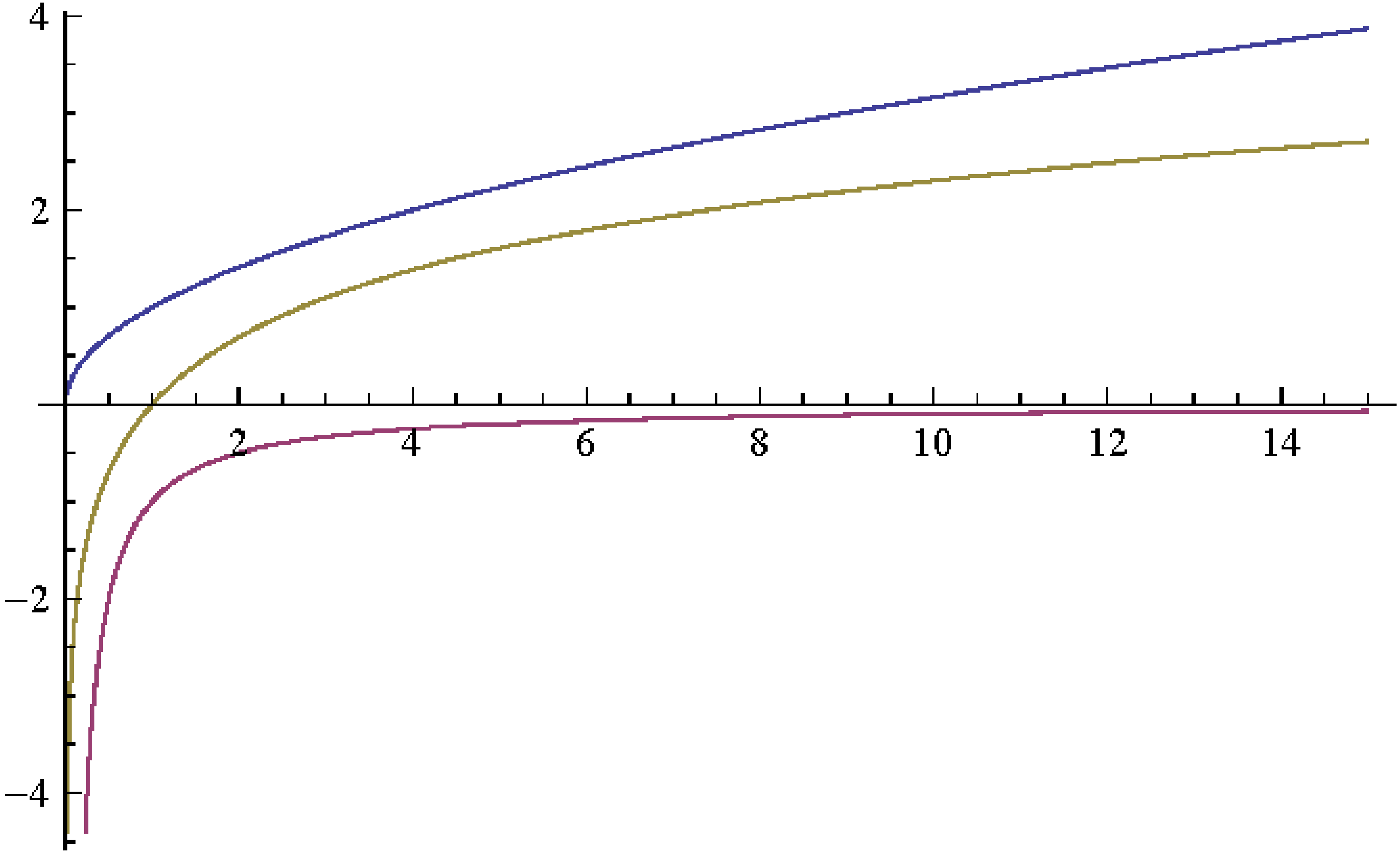
$$\lambda(E(W)) = -.5\sigma_W^2 \left.\frac{U''}{U'}\right|_{W=E(W))} = .5\sigma_W^2 R_A(E(W)).$$

- Here, $R_A(E(W))$ represents the *Arrow-Pratt Coefficient of Absolute Risk Aversion*, evaluated at $E(W)$.

Arrow-Pratt Risk Aversion Measures

- Note the following:
  - $R_A(E(W))$ equals -1 multiplied by the ratio of the second derivative of utility divided by marginal utility (both of which are evaluated at $W = E(W)$).
  - By measuring the *curvature* of the utility function, $R_A(W)$ provides a mathematical characterization of the consumer's degree of risk aversion.
  - Consequently, a given consumer's risk premium depends upon 1) her degree of risk aversion, and 2) the riskiness of the contemplated gamble or investment.

Different Risk Averse Utility Functions



Different Risk Averse Utility Functions

- Logarithmic utility: $U = \ln W$; note that marginal utility is positive; i.e., $dU / dW = 1 / W > 0$ and diminishing; i.e., $d^2U / dW^2 = -1 / W^2 < 0.$
- Power utility: $U = W^n$, where $0 < n < 1$; note that marginal utility is positive; i.e., $dU / dW = nW^{n-1} > 0$ and diminishing; i.e., $d^2U / dW^2 = (n-1)nW^{n-2} < 0.$
- $U = -W^{-1}$; note that marginal utility is positive; i.e., $dU / dW = -1\left(-W^{-2}\right) > 0$ and diminishing; i.e., $d^2U / dW^2 = dU / dW = 2\left(-W^{-3}\right) < 0.$

Arrow-Pratt Risk Aversion Measures

- Let's derive the Arrow-Pratt coefficients of absolute risk aversion for the logarithmic ($U_1 = \ln W$), power ($U_2 = W^n$, where $0 < n < 1$), and $U_3 = -W^{-1}$ functions.
  - $R_A^1(W) = 1/W$;
  - $R_A^2(W) = \frac{(1-n)}{W}$; and
  - $R_A^3(W) = \frac{2}{W}$.
- Therefore, U_3 is the most risk averse utility, U_1 is less risk averse than U_3 but more risk averse than U_2, and U_2 is the least risk averse.
- Also note that for all three functions, absolute risk aversion is decreasing in wealth! (*DARA*)

DARA Preferences

- Decreasing absolute risk aversion (*DARA*) implies that as one grows richer (poorer), she becomes less (more) risk averse.
- A person with *DARA* preferences increases (reduces) the dollar amount that she is willing to put at risk as she becomes wealthier (poorer). This seems intuitively plausible.

Arrow-Pratt Versus "Exact" Method

See the Arrow-Pratt Spreadsheet!

Risk Aversion Class Problem

Relative (Proportional) Risk Aversion

- $R_R(W) = WR_A(W)$.
- If $R_R(W)$ is decreasing/constant/increasing in W, then $U(W)$ displays decreasing/constant/increasing relative (proportional) risk aversion.
- The relative risk aversion coefficient indicates what proportion of your wealth that you are willing to put at risk, whereas absolute risk aversion measures the dollar amount.

Interpreting Arrow-Pratt risk aversion measures

- Numerous empirical studies have examined consumer behavior in various market settings; e.g., product markets, labor markets and financial markets.
- Generally, consumers exhibit decreasing absolute risk aversion (*DARA*) and constant relative risk aversion (*CRRA*).