Decision Making under Risk and Uncertainty (Part 4 of 4)

What we learned from the last lecture

- Using Arrow-Pratt to compare degree of risk aversion
- Broader Definitions of Risk and Risk Preference
- The game plan for today: Mean-Variance Analysis and Stochastic Dominance

EU Theory, *MV* Analysis, and *SD* Analysis

- Expected utility (*EU*) theory is the foundation for decision-making under risk and uncertainty.
- Although *EU* theory is elegant, a more practical, scientific (i.e., data-based) framework would be quite helpful as we seek to implement the *EU* model; i.e., can we find useful shortcuts to *EU*?
- Two shortcuts come to mind:
  - Mean-Variance (*MV*) Analysis
  - Stochastic Dominance (*SD*) Analysis

Mean-Variance (*MV*) Analysis

- *MV* analysis = *EU* analysis if variance is a "complete" risk measure and if:
  - $E(x) > E(y)$ and $\sigma_x^2 < \sigma_y^2$;
  - $E(x) > E(y)$ and $\sigma_x^2 = \sigma_y^2$; or
  - $E(x) = E(y)$ and $\sigma_x^2 < \sigma_y^2$. (mean preserving spread)
- However, if $E(x) > E(y)$ and $\sigma_x^2 > \sigma_y^2$, then *MV* analysis fails and we must rely on *EU* analysis!

MV Analysis Numerical Example

Consider two risky prospects, X_1 and X_2, with payoffs given by:

$$X_1 = \begin{cases} 1 \text{ with probability } \tfrac{1}{2} \\ 9 \text{ with probability } \tfrac{1}{2} \end{cases} \quad \text{and}$$

$$X_2 = \begin{cases} 4 \text{ with probability } \tfrac{99}{100} \\ 81 \text{ with probability } \tfrac{1}{100} \end{cases}$$

Assume that your initial wealth (W_0) is \$0, and your utility $U(W)$ $= \sqrt{W}$.

MV Analysis Numerical Example

- Which prospect is preferred according to the mean-variance rule?

SOLUTION: $E(W_1) = .5(1) + .5(9) = 5$ and

$$\sigma_{W_1} = \sqrt{.5(1-5)^2 + .5(9-5)^2} = 4, \text{ whereas}$$

$$E(W_2) = .99(4) + .01(81) = 4.76 \text{ and}$$

$$\sigma_{W_2} = \sqrt{.99(4-4.77)^2 + .01(81-4.77)^2} = 7.66.$$

According to the MV rule, risk 1 is preferred to risk 2 because and $E\left(W_1\right) > E\left(W_2\right)$ and $\sigma_{W_1} < \sigma_{W_2}$.

MV Analysis Example

- Note that $E(U(W_1)) = .5(1) + .5(3) = 2,$ and $E(U(W_2)) = .99(2) + .01(9) = 2.07; \therefore E(U(W_1)) < E(U(W_2)).$
- Why the apparent conflict between MV and EU?
  - Note that risk 2 is highly positively skewed (i.e., it provides a *small* chance of a really *large* payoff), whereas risk 1 is symmetric about its mean.
  - Apparently risk 2's positive skewness effect more than offsets its negative expected value, variance, and kurtosis effects.
  - Here, the MV rule is not appropriate because variance is not a "complete" risk measure. If one selected according to the mean variance rule, one would make the wrong choice!

Stochastic Dominance Analysis

- Dictionary definition of "stochastic":
  - Involving or containing a random variable or variables;
  - Involving chance or probability.
- Stochastic dominance analysis involves evaluating risks by comparing their probability distributions.

Stochastic Dominance Analysis

- Stochastic dominance (*SD*) Analysis provides an alternative (less restrictive) framework (compared with *MV*).
  - *SD* makes it possible to evaluate a broader set of risks than the *MV* rule.
  - Stochastic Dominance evaluates risks independently of the specific trade-offs (between expected value, standard deviation, skewness, kurtosis, etc.) represented by an agent's utility function.

First Order Stochastic Dominance

- Consider two cumulative distribution functions of wealth ($F(W)$ and $G(W)$) over some closed interval $I = [W_{\min}, W_{\max}]$.
- Then $F(W)$ First Order Stochastic Dominates ($FOSD$) $G(W)$ if $G(W) \geq F(W)$ for all W between $W_{\min}$ and $W_{\max}$ (with the inequality holding for some W).
- An important implication of F $FOSD$ G is that

$$F\ FOSD\ G \rightarrow E_{\mathrm{F}}[U(W)] > E_{\mathrm{G}}[U(W)].$$

First Order Stochastic Dominance

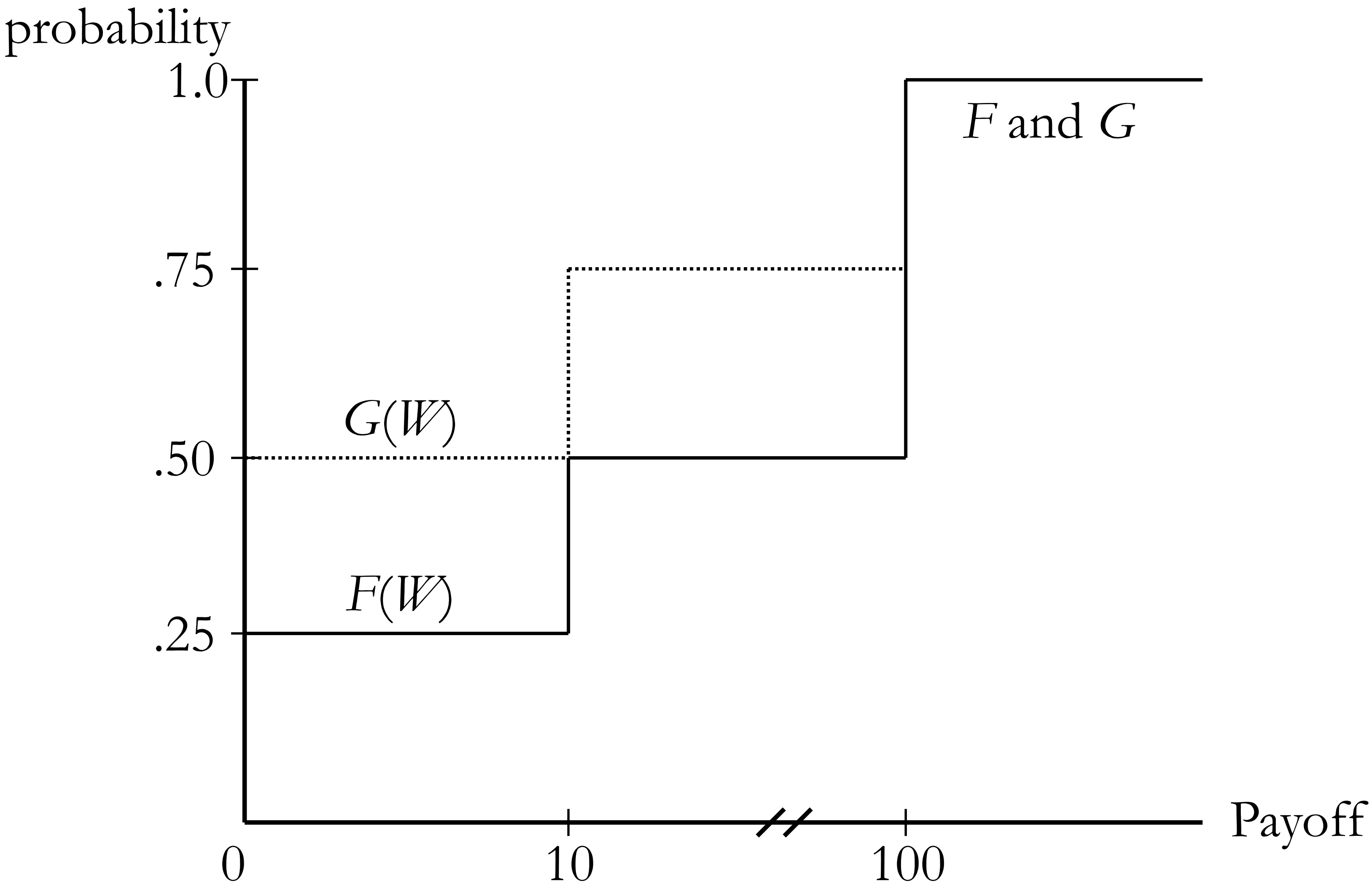
- Example of *FOSD*: consider 2 assets with 3 payoffs:

$W(s)$	\$0	\$10	\$100
$f(W(s))$	0.25	0.25	0.5
$g(W(s))$	0.5	0.25	0.25

- The cumulative probability functions are:

$W(s)$	\$0	\$10	\$100
$F(W(s))$	0.25	0.5	1
$G(W(s))$	0.5	0.75	1

First Order Stochastic Dominance



Second Order Stochastic Dominance

- First Order Stochastic Dominance is a very strong condition; Second Order Stochastic Dominance provides a less restrictive definition of dominance:
- $F(W)$ Second Order Stochastic Dominates ($SOSD$) $G(W)$ if

$$\sum_{s=1}^{n}\left(G(W_s)-F(W_s)\right)>0.$$

- An important implication of F $SOSD$ G is that

$$F\ SOSD\ G \rightarrow E_{\mathrm{F}}[U(W)] > E_{\mathrm{G}}[U(W)].$$

- Also note that if F $FOSD$ G, then F $SOSD$ G.
- The only case in which the equality holds is in the case of the mean preserving spread (more on this later).

Second Order Stochastic Dominance

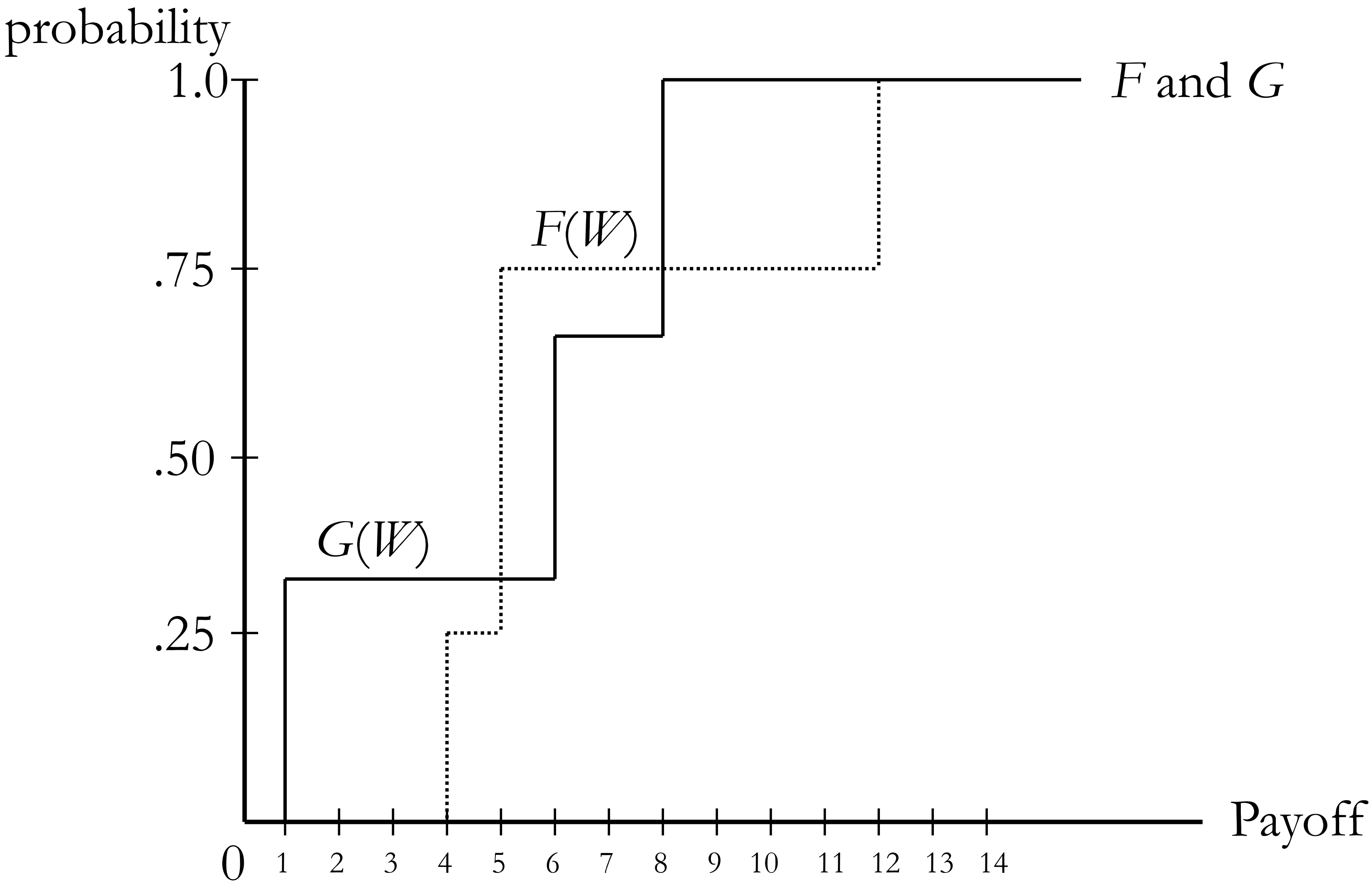
- Suppose there are two assets, asset F and asset G, that provide the following set of risky payoffs:

Asset F			Asset G	
$W(s)$	$f(W(s))$		$W(s)$	$g(W(s))$
4	25%		1	33%
5	50%		6	33%
12	25%		8	33%

- The cumulative probability functions for F and G are:

Asset F			Asset G	
$W(s)$	$F(W(s))$		$W(s)$	$G(W(s))$
4	25%		1	33%
5	75%		6	67%
12	100%		8	100%

Second Order Stochastic Dominance



Second Order Stochastic Dominance

- There is no *FOSD*, since the cumulative distribution functions "cross over" at payoffs of \$5 and \$8.
- Is there *SOSD*?

W_s	$f(W_s)$	$F(W_s)$	$g(W_s)$	$G(W_s)$	$G(W_s)$ - $F(W_s)$
1	0.00%	0.00%	33.33%	33.33%	33.33%
4	25.00%	25.00%	0.00%	33.33%	8.33%
5	50.00%	75.00%	0.00%	33.33%	-41.67%
6	0.00%	75.00%	33.33%	66.67%	-8.33%
8	0.00%	75.00%	33.33%	100.00%	25.00%
12	25.00%	100.00%	0.00%	100.00%	0.00%
				$\Sigma(G(W_s)$-$F(W_s))$	**16.67%**