Portfolio Theory Class Problem

Finance 4335

Consider a market with only two securities, numbered 1 and 2. Expected returns are $E(r_1) = 12\%$ and $E(r_2) = 8\%$, standard deviations are $\sigma_1 = 10\%$ and $\sigma_2 = 4\%$, and $\rho_{12} = -1$; i.e., returns are perfectly negatively correlated.

A. Let $w_1 \geq 0$ be the proportion of wealth invested in security 1 and $w_2 \geq 0$ be the proportion of wealth invested in security 2, where $w_1 + w_2 = 1$. Given these portfolio allocation constraints, what is the range of expected portfolio returns for all possible portfolio combinations consisting of these two securities (including cases where $w_1 = 0$ or $w_2 = 0$)?

SOLUTION: When a portfolio consists of only two securities, then $E(r_p) = w_1 E(r_1) + w_2 E(r_2)$; since $E(r_1) = 12\%$ and $E(r_2) = 8\%$, it follows that $E(r_p) = w_1(.12) + w_2(.08)$ for all possible portfolio combinations of securities 1 and 2. Thus, $8\% \leq E(r_p) \leq 12\%$. The lower boundary obtains when $w_1 = 0$ and $w_2 = 1$, whereas the upper boundary obtains when $w_2 = 0$ and $w_1 = 1$.

B. What is the range of standard deviations for all possible portfolio combinations considered in Part A of this problem (including cases where $w_1 = 0$ or $w_2 = 0$)?

SOLUTION: For $n = 2$, portfolio standard deviation is $\sigma_p = \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2}$, and when $\rho_{12} = -1$, this simplifies to $\sigma_p = \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 - 2w_1w_2\sigma_1\sigma_2}$. For these two securities, $\sigma_p = \sqrt{w_1^2(.1^2) + w_2^2(.04^2) - w_1w_2(.08)}$.

When $\rho_{12} = -1$, the minimum variance portfolio will be riskless since a perfect negative correlation implies that a weighting scheme exists such that the risks of both securities offset each other perfectly. Thus, the range of standard deviations for all portfolio combinations of securities 1 and 2 is $0\% \leq \sigma_p \leq 10\%$. The zero percent lower boundary is provided by the minimum variance portfolio, and the 10% upper boundary obtains when there is 100% investment in security 1.

C. What is the expected return and standard deviation for the minimum risk combination of securities 1 and 2?

SOLUTION: The weights for the minimum risk combination of 2 securities can be determined using the following equations: $w_1 = \dfrac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}$ and $w_2 = 1 - w_1$. Therefore, $w_1 = \dfrac{16 + 40}{100 + 16 + 80} = \dfrac{56}{196} = \dfrac{2}{7}$, and $w_2 = 1 - w_1 = \dfrac{5}{7}$. Consequently,

$E(r_p) = w_1(.12) + w_2(.08) = (2/7)(.12) + (5/7)(.08) = 9\,{}^1\!/_7\%$, and

$\sigma_p = \sqrt{w_1^2(.1^2) + w_2^2(.04^2) - w_1w_2(.08)} = \sqrt{\frac{4}{49}(.01) + \frac{25}{49}(.016) - \frac{10}{49}(.08)} = 0.$

D. Suppose the riskless lending and borrowing rate is 10%. Describe a trading strategy involving security 1, security 2, and the riskless asset which would enable you to earn riskless arbitrage profits without investing any of your own money.

SOLUTION: Since the riskless asset returns 10% whereas the minimum (zero) risk combination of securities 1 and 2 only returns $9\frac{1}{7}\%$, we have an opportunity for riskless arbitrage. Specifically, we would want to invest the proceeds of a short sale of the minimum risk combination of securities 1 and 2 and use it to fund a purchase of the riskless asset. The net result of such a trade is that we would earn positive returns with zero risk and zero net investment.

E. Describe how and why competition amongst investors will cause the returns on these three assets to adjust, such that the opportunity for riskless arbitrage ceases to exist.

SOLUTION: A riskless arbitrage opportunity like the one described in part C would result in excess demand for the riskless asset and excess supply of minimum risk combinations of securities 1 and 2. In response to such mispricing, investors sell securities 1 and 2 and buy the riskless asset, thereby bidding down the prices for securities 1 and 2 and bidding up the price of the riskless asset. These price changes increase the prospective returns on securities 1 and 2 while decreasing the prospective return on the riskless asset, until it is no longer possible to earn positive riskless arbitrage profits, since the prospective return on the minimum variance portfolio consisting of securities 1 and 2 will at that point be equal to the prospective return on the riskless asset.