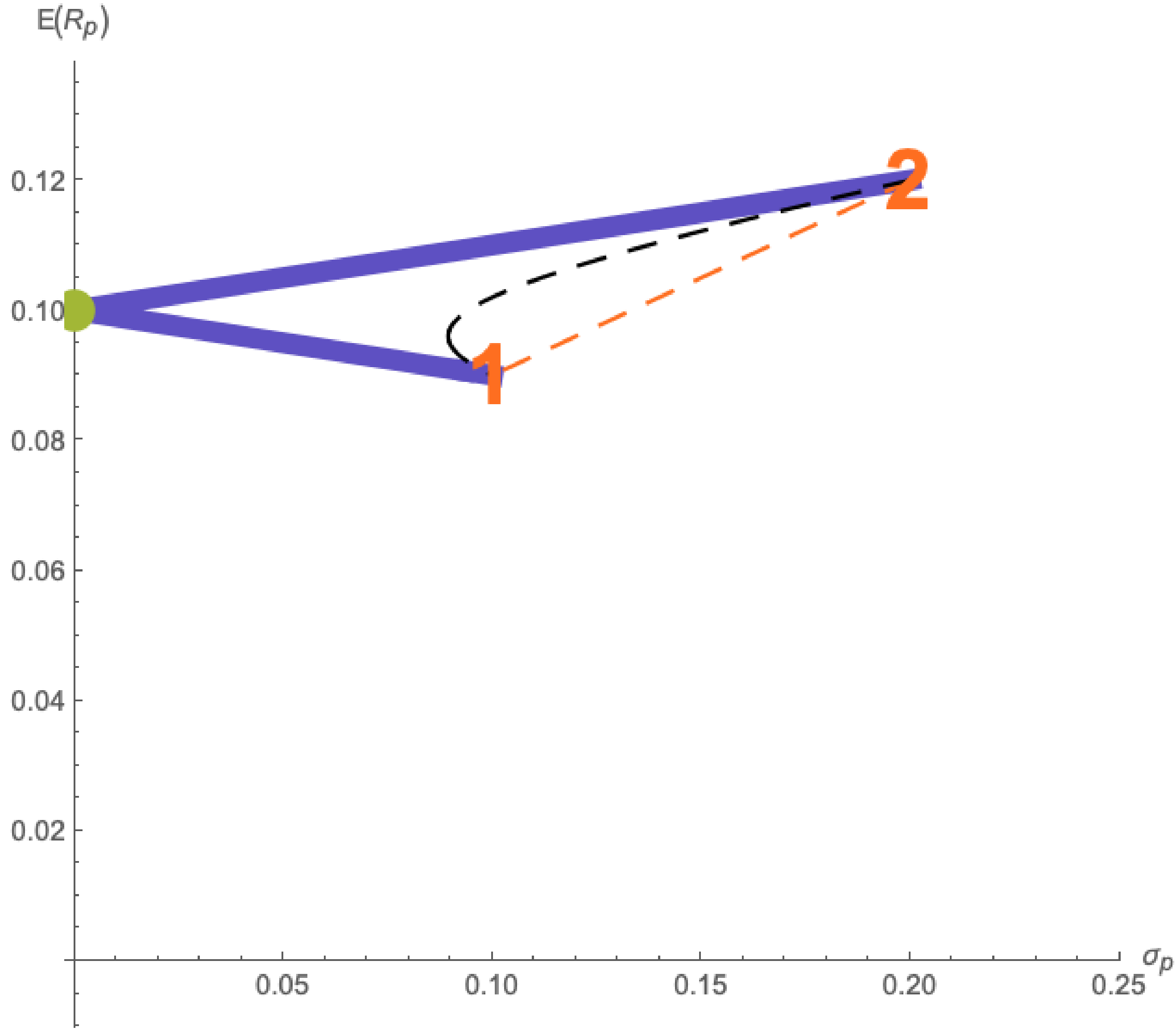
Portfolio and Capital Market Theory

- **Two-asset portfolio:** Given $E(r_1)$, $E(r_2)$, σ_1, σ_2, and $\sigma_{1,2}$, determine the *feasible* set of portfolios in $\{E(r_p), \sigma_p\}$ space by selecting arbitrary combinations of w_1 and $w_2 = 1 - w_1$, where $E(r_p) = w_1 E(r_1) + w_2 E(r_2)$ and $\sigma_p = \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12}}$.
- Next, identify that subset of portfolios which are *mean-variance efficient* (MVE); the end-point of the "efficient frontier" is the minimum variance portfolio (MVP), where $w_1 = \dfrac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}$.
- For example, suppose $E(r_1) = .09$, $E(r_2) = .12$, $\sigma_1 = .10$, $\sigma_2 = .20$; then $\sigma_{1,2} = -.02$, $w_1 = \frac{.04-(-.02)}{.01+.02-2(-.02)} = 2/3$, $E(r_p) = .10$, and $\sigma_p = 0$. Then the efficient frontier consists of all portfolios where $w_1 \leq 2/3$, while all portfolios where $w_1 > 2/3$ are *mean variance inefficient.*
- See http://fin4335.garven.com/spring2024/2assetportfolio.xlsx.

Mean-variance efficiency with 2 risky assets



Mean-variance efficiency with n risky securities

- Given $E(r_i)$, σ_i and $\sigma_{i,j}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, n$, determine the *feasible* set of portfolios in $\{E(r_p), \sigma_p\}$ space by selecting arbitrary combinations of $w_1, w_2, \ldots, w_n$, where $\sum_{i=1}^{n} w_i = 1$, $E(r_p) = \sum_{i=1}^{n} w_i E(r_i)$ and $\sigma_p = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}}$.

- Next, identify that subset of portfolios which are *mean-variance efficient* (MVE); as in the two-asset case, the end point of the "efficient frontier" in the n-asset case is the minimum variance portfolio (MVP).

- All portfolios on the northwest perimeter of the feasible set of portfolios, beginning with the MVP, are located on the efficient frontier.

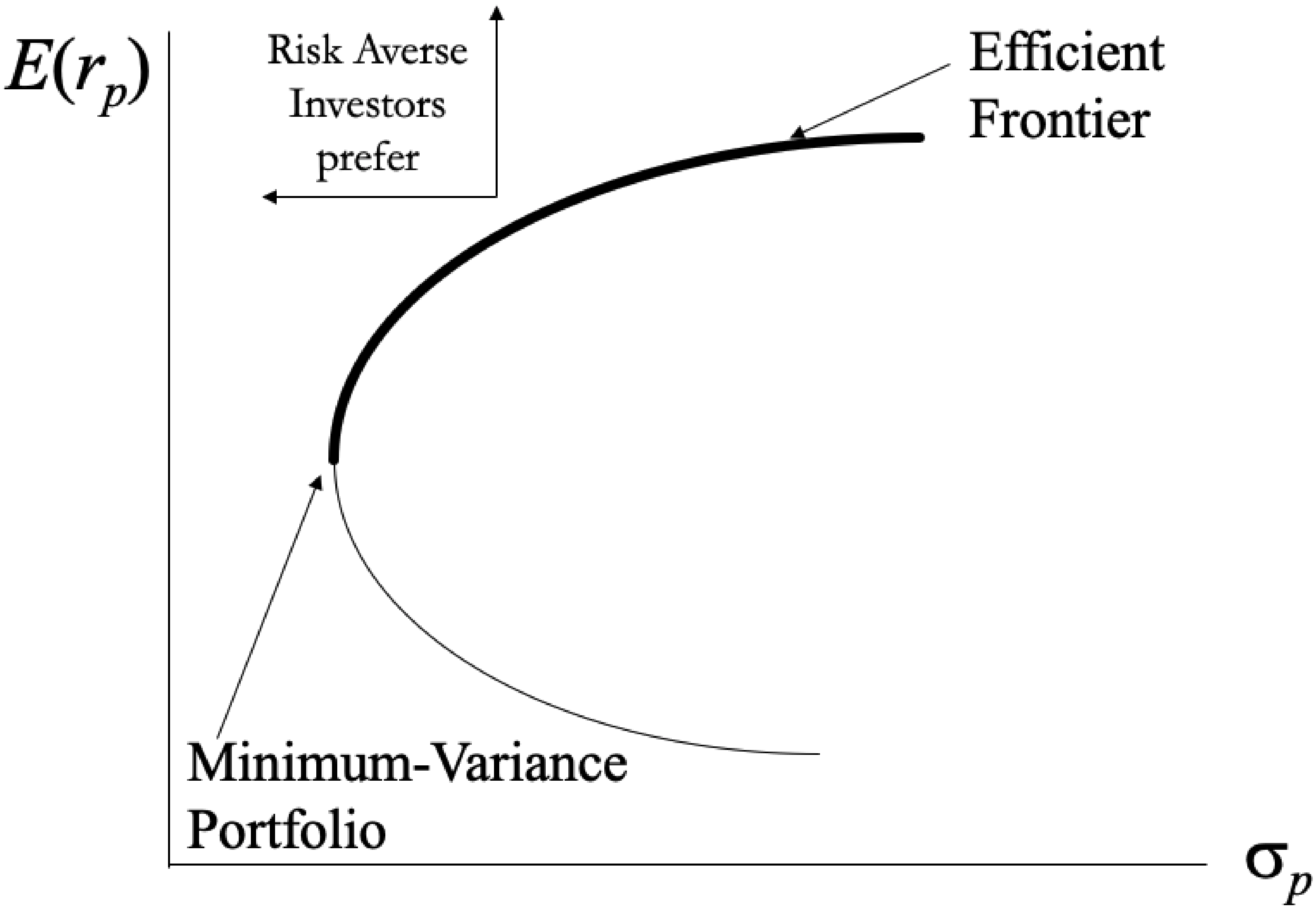
Mean-variance efficiency with n risky securities

$$\underset{\{w_1,w_2,\ldots,w_n\}}{\text{Minimize}} \;\; \sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij},$$

subject to $E(r_p) = \sum_{i=1}^{n} w_i E(r_i) = \chi$ and $\sum_{i=1}^{n} w_i = 1$.

- The analyst traces out the efficient frontier for a set of n assets by iteratively solving this optimization problem.
- The $E(r_p)$ constraint is initially nonbinding, since the analyst must first determine the *unique* asset allocation scheme for the minimum variance portfolio (MVP).
- Next, the analyst calculates $E(r_{mvp})$ and determines the rest of the efficient frontier by iteratively calculating asset allocation schemes that produce increasingly higher values for χ (and consequently higher values for both $E(r_p)$ and σ_p^2).

Mean-variance efficiency with n risky securities



Optimal portfolio selection

- Next, we turn our attention to the issue of how to select an optimal portfolio. This requires revisiting the Arrow-Pratt framework.

- According to Arrow-Pratt, $W_{CE} = E(W) - \lambda$, where $\lambda = .5\sigma_W^2 R_A(E(W))$, $R_A(W) = -U''/U'$ is the *Arrow-Pratt measure of absolute risk aversion*, and $R_A(E(W))$ represents the degree of absolute risk aversion for a given $E(W)$.

- Absolute risk aversion corresponds to the *dollar amount* of wealth that an investor is willing to put at risk, whereas *relative risk aversion* $R_R = WR_A(W)$ corresponds to the *proportion* of wealth that an investor is willing to put at risk.

- Risk tolerance (τ) is the reciprocal of relative risk aversion; i.e., $\tau = 1/R_R$.

Optimal portfolio selection

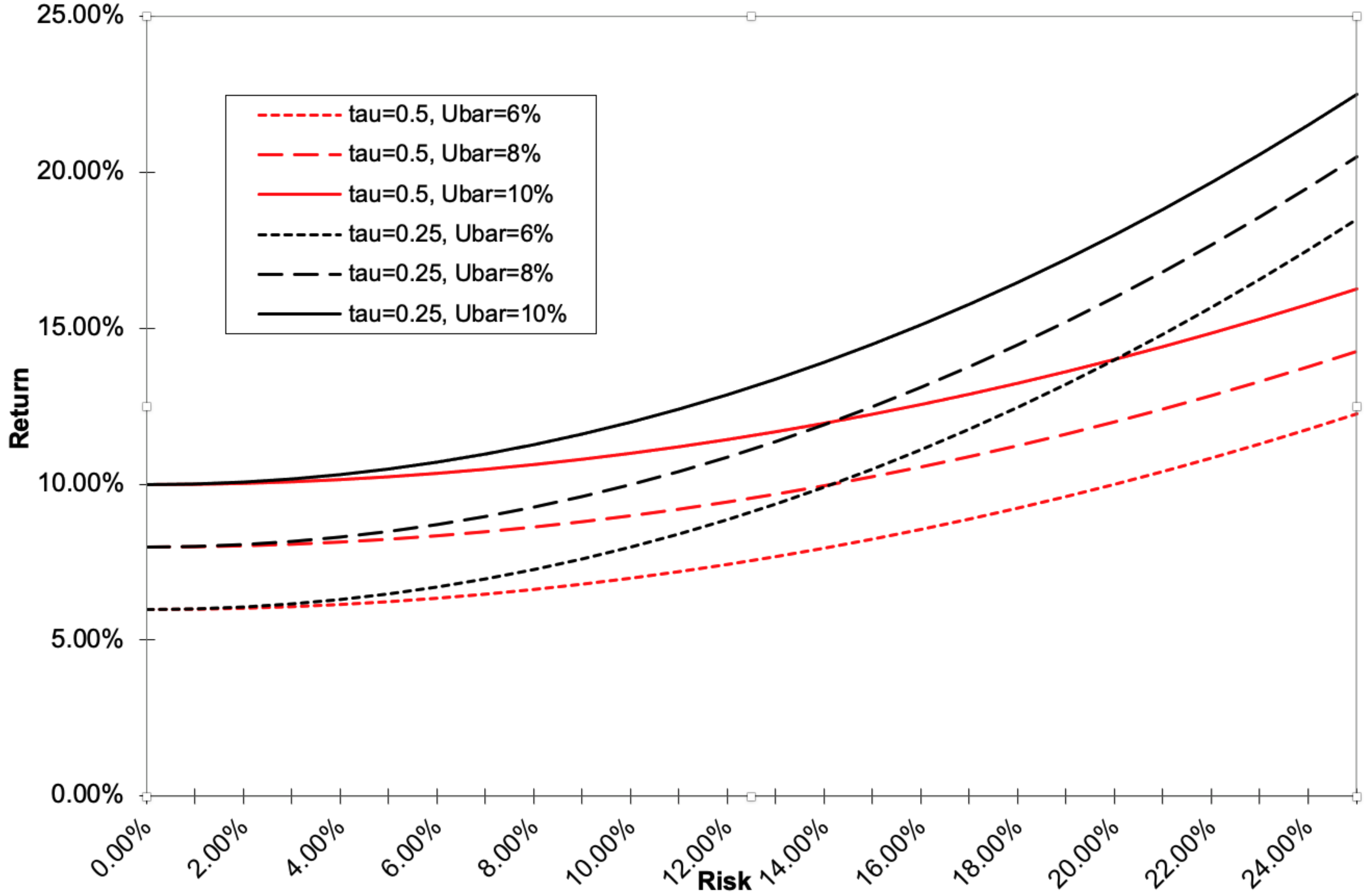
- Since $W_{CE} = E(W) - \lambda$, the certainty-equivalent of the *percentage change in wealth* (r_p^c) equals the difference between the expected return on the investor's portfolio ($E(r_p)$) minus the (percentage) risk premium; i.e., $.5\sigma_p^2 R_R = .5\sigma_p^2/\tau$. Thus, $r_p^c = E(r_p) - .5\sigma_p^2/\tau \Rightarrow E(r_p) = r_p^c + .5\sigma_p^2/\tau$.

- Since maximizing expected utility is equivalent to maximizing the certainty-equivalent portfolio return, $E(r_p) = r_p^c + .5\sigma_p^2/\tau$ is our indifference curve equation.

- The investor's utility is constant along each possible the indifference curve. The *higher* the risk tolerance τ, the *flatter* the curve.

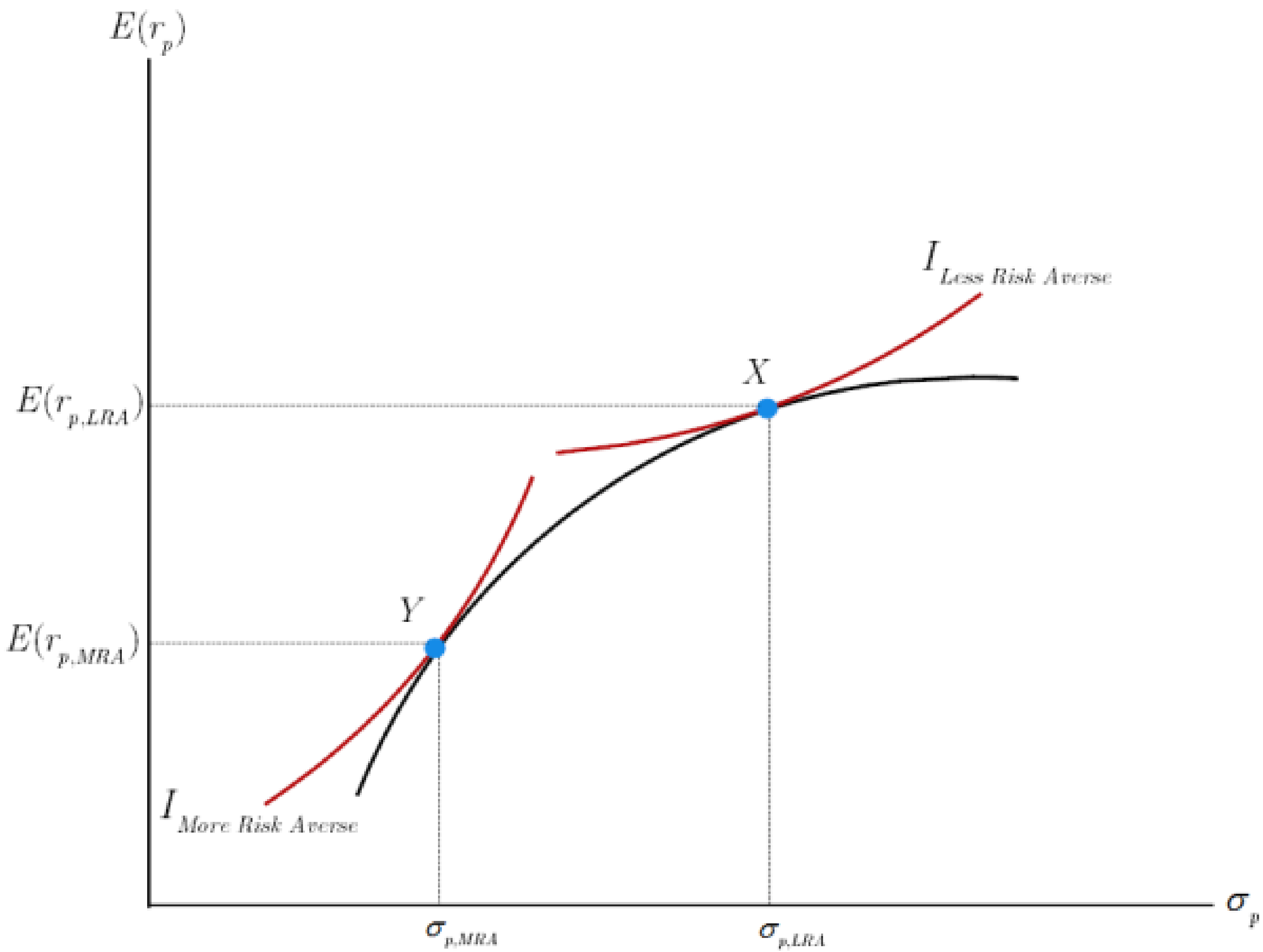
- On page 8, we show indifference curves for $\tau = .25$ compared with $\tau = .50$). We also vary r_p^c from 6% to 10%.

Optimal portfolio selection



Optimal portfolio selection

The expected utility maximizing investor finds the portfolio where her highest indifference curve is tangent to the efficient frontier.

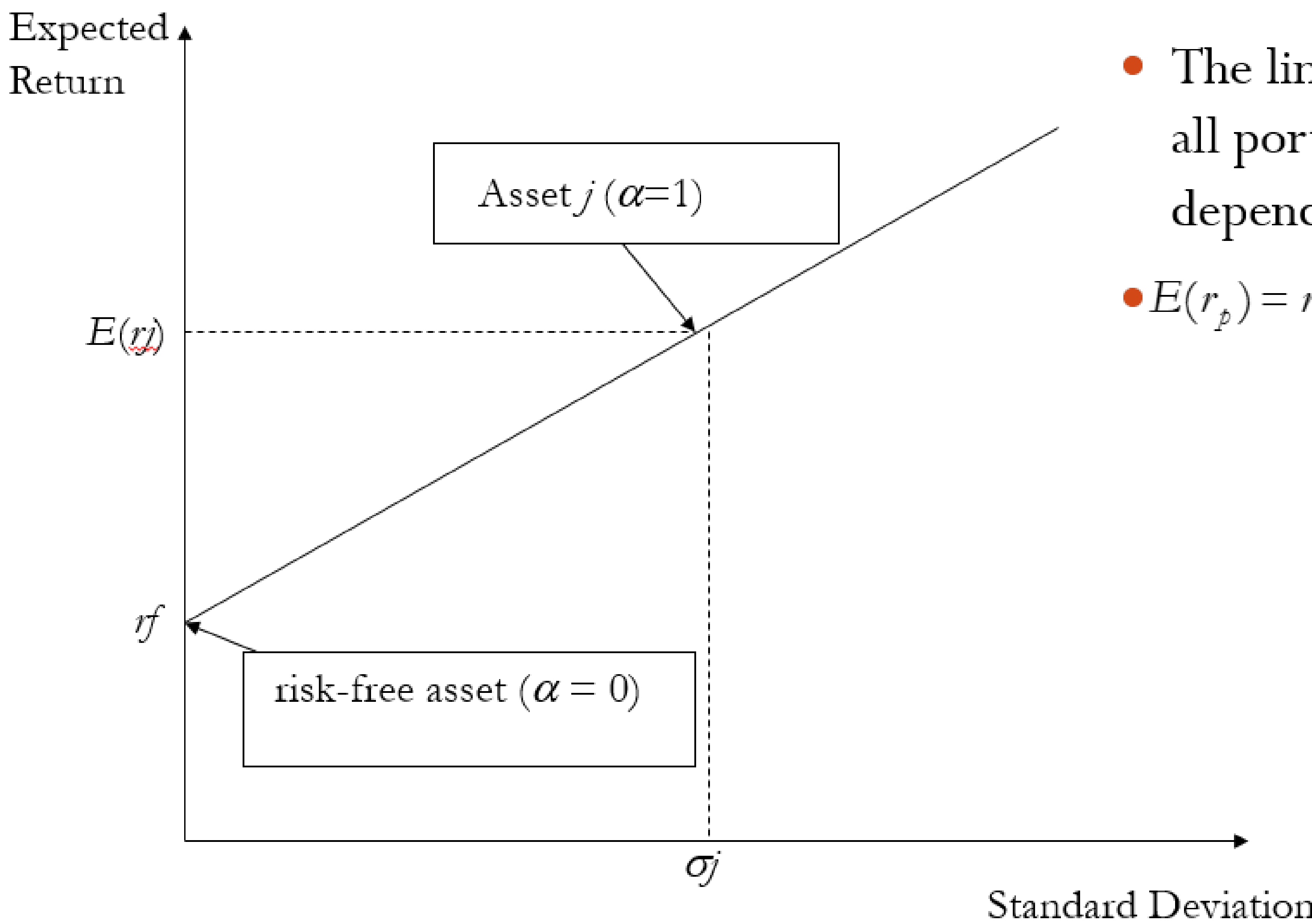


Optimal portfolio selection with a riskless security

- Suppose the investor limits her portfolio selection to a riskless security with expected return r_f and zero standard deviation and a risky security (or portfolio of risky securities) with expected return $E(r_j)$ and standard deviation σ_j.
- Let α denote the proportion of the portfolio invested in the risky security. Thus, the expected return $E(r_p)$ and standard deviation σ_p for the portfolio are: $E(r_p) = \alpha E(r_j) + (1-\alpha)\, r_f$, and $\sigma_p = \sqrt{\alpha^2\sigma_j^2 + (1-\alpha)^2\sigma_f^2 + 2\alpha(1-\alpha)\sigma_{j,f}} = \alpha\sigma_j$.
- Since $\alpha = \sigma_p/\sigma_j$, we replace α in the equation for $E(r_p)$ with the ratio σ_p/σ_j, yielding

$$E(r_p) = r_f + \frac{E(r_j) - r_f}{\sigma_j}\sigma_p.$$

Optimal portfolio selection with a riskless security



- The line represents all portfolios depending on α
- $E(r_p) = r_f + \frac{E(r_j) - r_f}{\sigma_j} \sigma_p$

Optimal portfolio selection with a riskless security

- From page 7, since $r_p^c = E(r_p) - .5\sigma_p^2/\tau$, $E(r_p) = \alpha E(r_j) + (1-\alpha)\, r_f$, and $\sigma_p^2 = \alpha^2\sigma_j^2$, it follows that

$$r_p^c = \alpha E(r_j) + (1-\alpha)r_f - (.5/\tau)\alpha^2\sigma_j^2.$$

- Differentiating this equation with respect to the α and setting the resulting expression equal to zero yields the first order condition:

$$dr_p^c/d\alpha = E(r_j) - r_f - (1/\tau)\alpha\sigma_j^2 = 0.$$

- Rearranging the first order condition and solving for α results in the following equation for α:

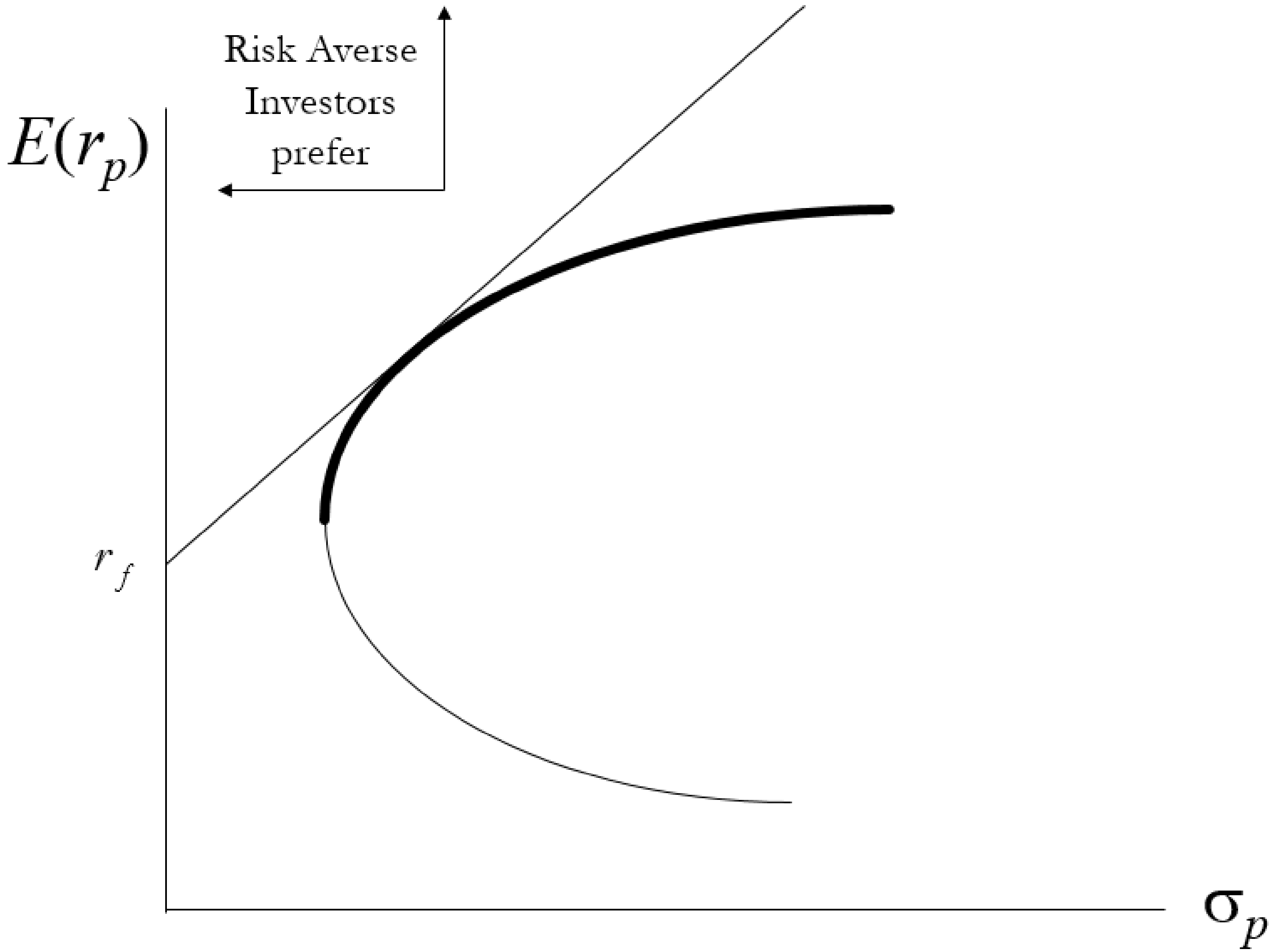
$$\alpha = \frac{(E(r_j) - r_f)}{\sigma_j^2}\tau = \frac{(E(r_j) - r_f)}{\sigma_j}\frac{\tau}{\sigma_j}.$$

Optimal portfolio selection with a riskless security

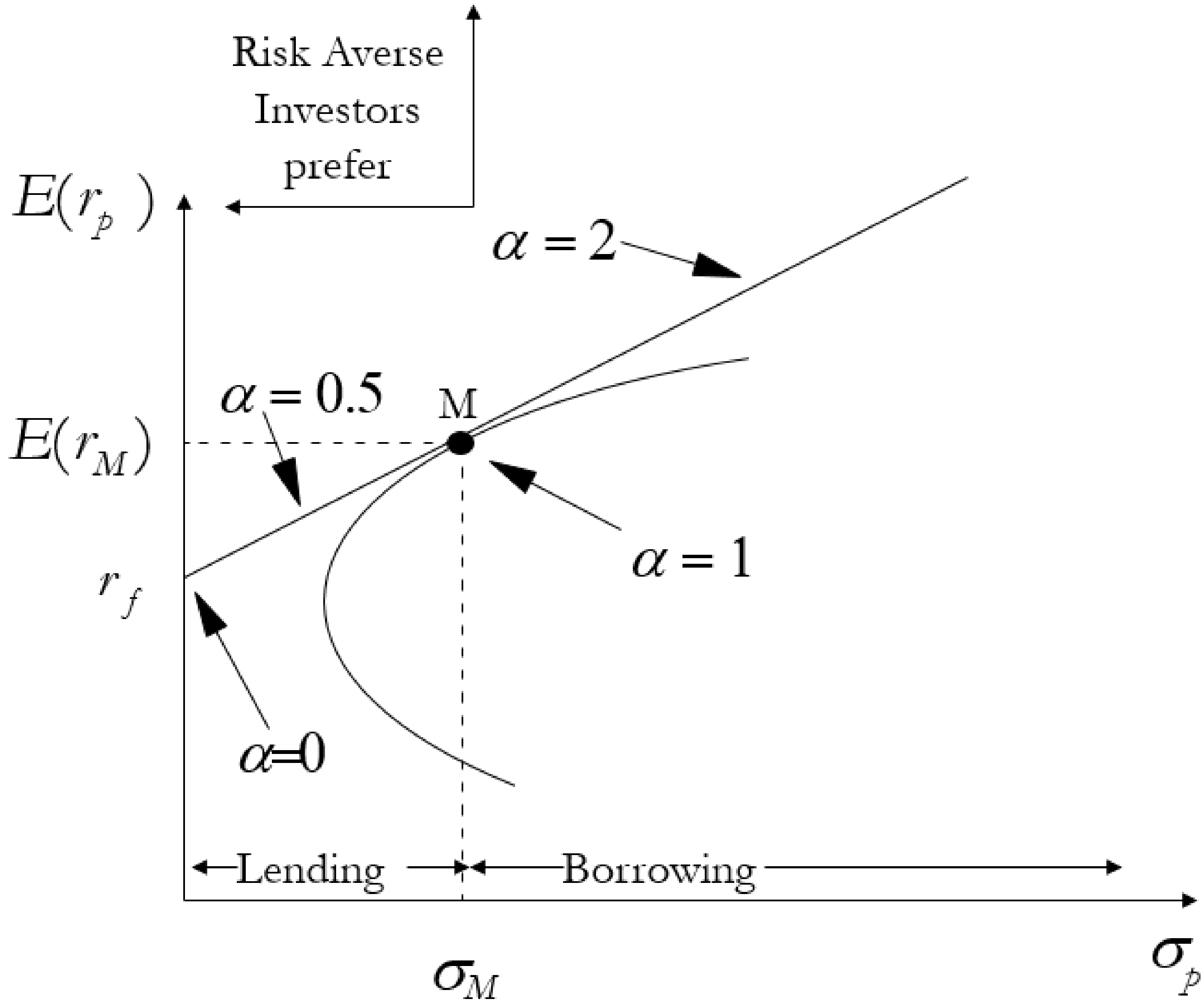
- In the $\alpha = \dfrac{(E(r_j) - r_f)}{\sigma_j}\dfrac{\tau}{\sigma_j}$ equation, the first ratio is the well-known "Sharpe Ratio"; thus, α is positively related to the Sharpe Ratio and τ, and inversely related to σ_j.
- Suppose $E(r_j) = 12\%$, $r_f = 4\%$, and $\sigma_j = 20\%$; the following table shows how changes in risk tolerance affect α:

Risk tolerance (τ)	α	1-α	$E(r_p)$	σ_p
1.0	200%	-100%	20%	40%
0.8	160%	-60%	17%	32%
0.6	120%	-20%	14%	24%
0.4	80%	20%	10%	16%
0.2	40%	60%	7%	8%
0.0	0%	100%	4%	0%

Optimal portfolio selection with a riskless security



Optimal portfolio selection with a riskless security



The Capital Market Line (CML)

- The line in the figure on page 15 is commonly referred to as the *capital market line* (CML). This is the efficient frontier in a world in which investors can borrow and lend money at the risk-free rate of interest. The equation for the Capital Market Line is:

$$E(r_p) = r_f + \frac{E(r_j) - r_f}{\sigma_M} \sigma_p.$$

- The expected rate of return on a mean-variance efficient portfolio consists of two components: 1) the return on a riskless security which compensates investors for the time value of money and 2) a risk premium which compensates investors for bearing risk.

The Capital Market Line (CML)

- An important implication of the Capital Market Line is that in equilibrium, all risk-return tradeoffs must be equal.
- Assume that the market portfolio consists of all (N) securities in the economy, and security j accounts for w_j percent of the market portfolio. Then the equations for the expected return $(E(r_M))$ and the variance (σ_M^2) are

$$E(r_M) = \sum_{j=1}^{N} w_j(E(r_j) - r_f) + r_f, \text{and}$$

$$\sigma_M^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{i,j}.$$

The Capital Market Line (CML)

- Suppose that we marginally increase w_j. Then $E(r_p)$ changes by $E(r_j) - r_f$, and σ_p^2 changes by $\sum_{i=1}^{N} w_i \sigma_{j,i} = \sigma_{j,M}$. Thus, the return/risk trade-off is $(E(r_j) - r_f)/\sigma_{j,M}$.
- In equilibrium, the risk-return tradeoff must be the same for all securities; i.e., $(E(r_i) - r_f)/\sigma_{i,M}) = (E(r_j) - r_f)/\sigma_{j,M})$ for all i and j. Therefore, if $(E(r_i) - r_f)/\sigma_{i,M}) \neq (E(r_j) - r_f)/\sigma_{j,M})$, then there is an *arbitrage* opportunity.
- Suppose $(E(r_i) - r_f)/\sigma_{i,M}) > (E(r_j) - r_f)/\sigma_{j,M})$. Then i offers a better risk-return trade-off than $j \Rightarrow$ investors buy i and short (sell) j. Consequently, in equilibrium, the risk-return trade-off must be equal for all securities; i.e., $(E(r_i) - r_f)/Cov(r_i, r_M)$ $= (E(r_j) - r_f)/\sigma_{j,M})$ for all i and j.

The Capital Asset Pricing Model (CAPM)

- If the risk-return tradeoff is the same for all i and j, than it must also be same for the market as it is for i and j; thus,

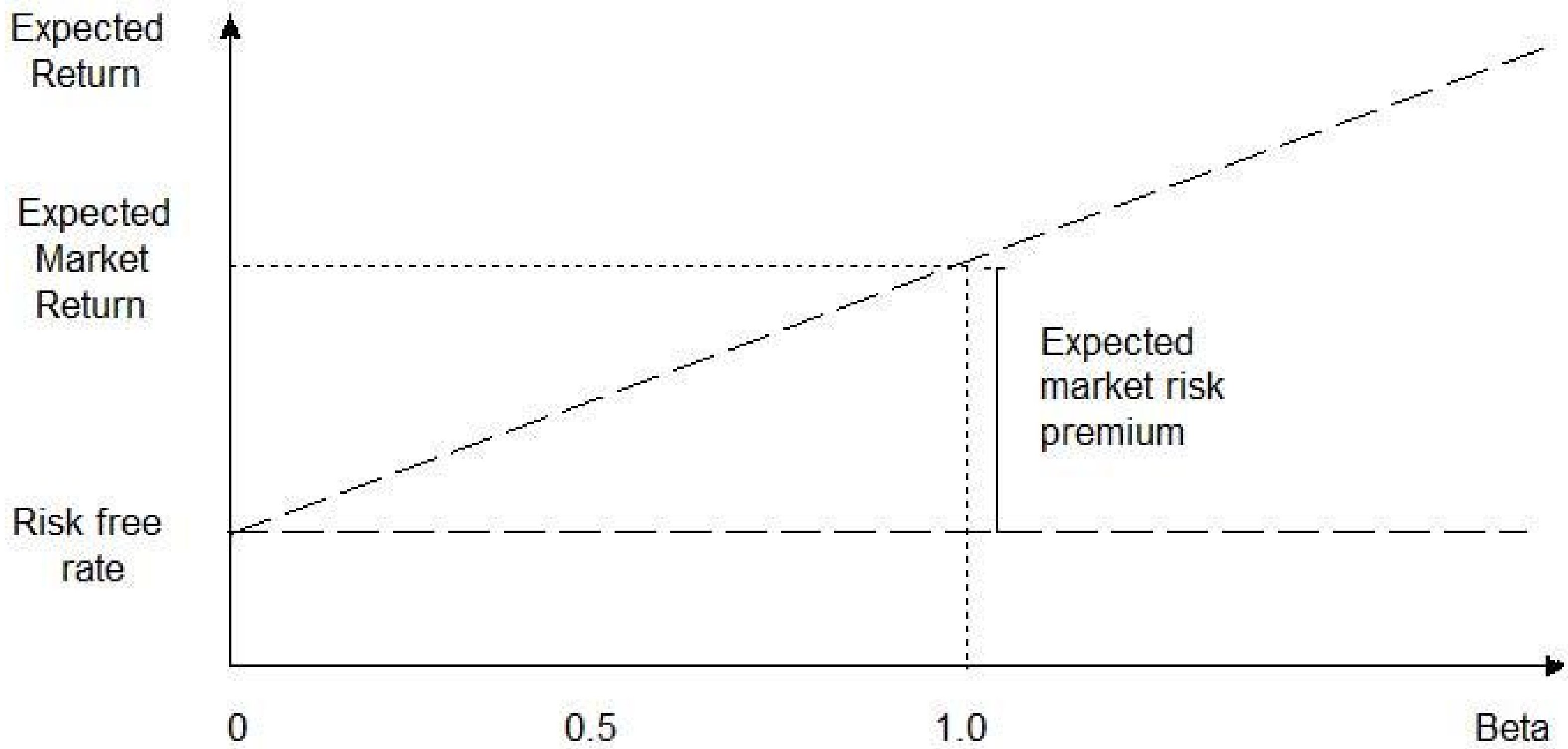
$$\frac{E(r_i) - r_f}{\sigma_{i,M}} = \frac{E(r_M) - r_f}{\sigma_M^2}.$$

- Next, solve this equation for $E(r_i)$:

$$E(r_i) = r_f + \frac{\sigma_{i,M}}{\sigma_M^2}(E(r_M) - r_f) = r_f + \beta_i(E(r_M) - r_f),$$

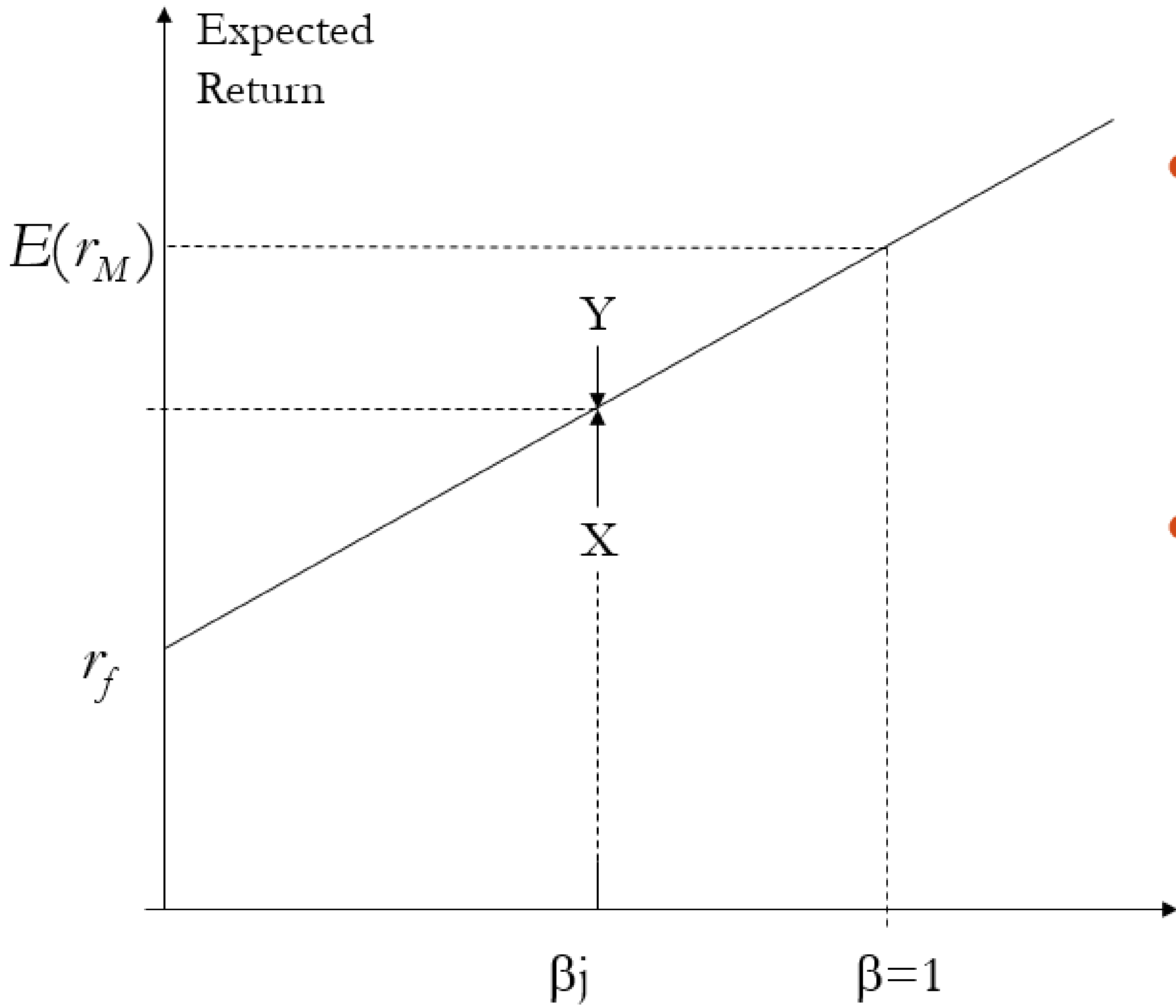
where $\beta_i = \frac{\sigma_{i,M}}{\sigma_M^2}$. This equation is commonly known as the *capital asset pricing model* or CAPM and is represented graphically in the figure on the next page.

The Capital Asset Pricing Model (CAPM)



- According to the CAPM equation $E(r_i) = r_f + \beta_i(E(r_M) - r_f)$, the expected rate of return on a risky security consists of 1) the return on a riskless security, and 2) a risk premium which is proportional to "beta", which measures "systematic" risk.

The Capital Asset Pricing Model (CAPM)



- Security j is *overvalued* at X:
  - price drops,
  - expected return rises.
- At Y, security j would be *undervalued*!
  - expected return falls
  - price increases

The Capital Asset Pricing Model (CAPM)

- The appropriate measure of risk for an individual security is its **beta**.
- Beta measures the sensitivity of the security to **market risk** factors.
- The higher the beta, the more sensitive the security is to market movements.
- The average security has a beta of 1.0.
- Portfolio betas are weighted averages of the betas for the individual securities in the portfolio.
- The **market risk premium** is $[E(r_M)\text{-}r_f]$.