CHAPTER 6

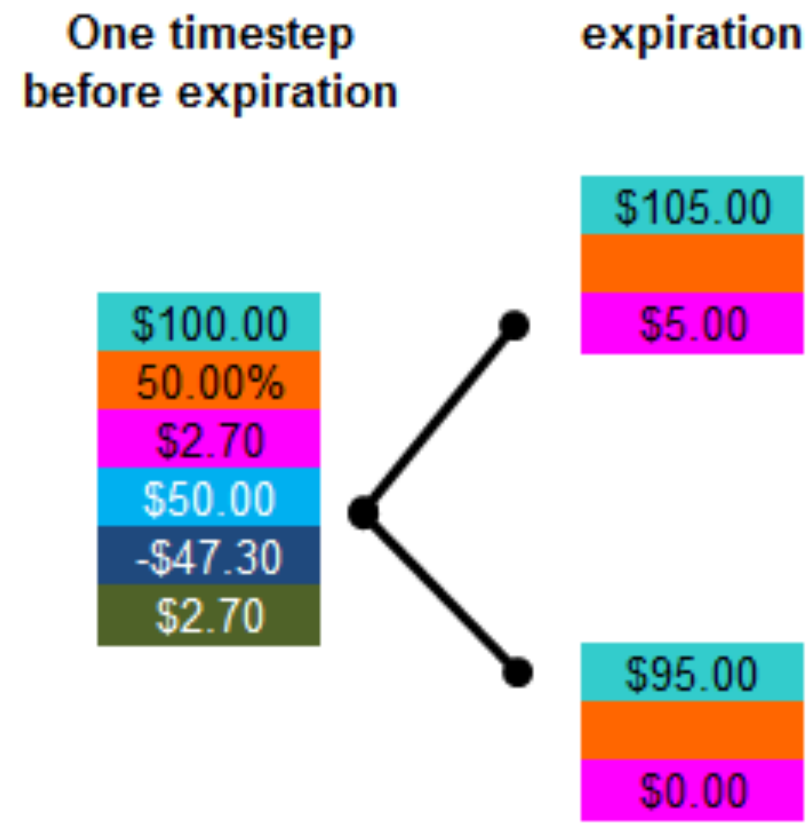
Derivatives Theory, part 2

Binomial Tree for a One-Step Call Option

Risk Neutral Valuation & Replicating Portfolio Approaches to Pricing a Call Option (One Timestep)

Asset	$100
u	1.05
d	0.95
q	0.5418
dt	0.0833
Interest rate	5%
Discount factor	0.9958
Strike	$100

Risk Neutral Valuation	Stock
	Delta
	Option
	Long Stock
Replicating Portfolio	Borrowing
	Net Value



Binomial Tree for a One-Step Put Option

Risk Neutral Valuation & Replicating Portfolio Approaches to Pricing a Put Option (One Timestep)

Asset	$100	**One timestep before expiration**	**expiration**
u	1.05		
d	0.95		
q	0.5418		$105.00
dt	0.0833		
Interest rate	5%	$100.00	$0.00
Discount factor	0.9958	-50.00%	
Strike	$100	$2.28	
		-$50.00	
Risk Neutral Valuation	Stock	$52.28	
	Delta	$2.28	
	Option		$95.00
	Short Stock		
Replicating Portfolio	Lending		$5.00
	Net Value		

Put-Call Parity

European options cannot be exercised prior to the expiration date; therefore, the portfolios must have identical values today; i.e.,

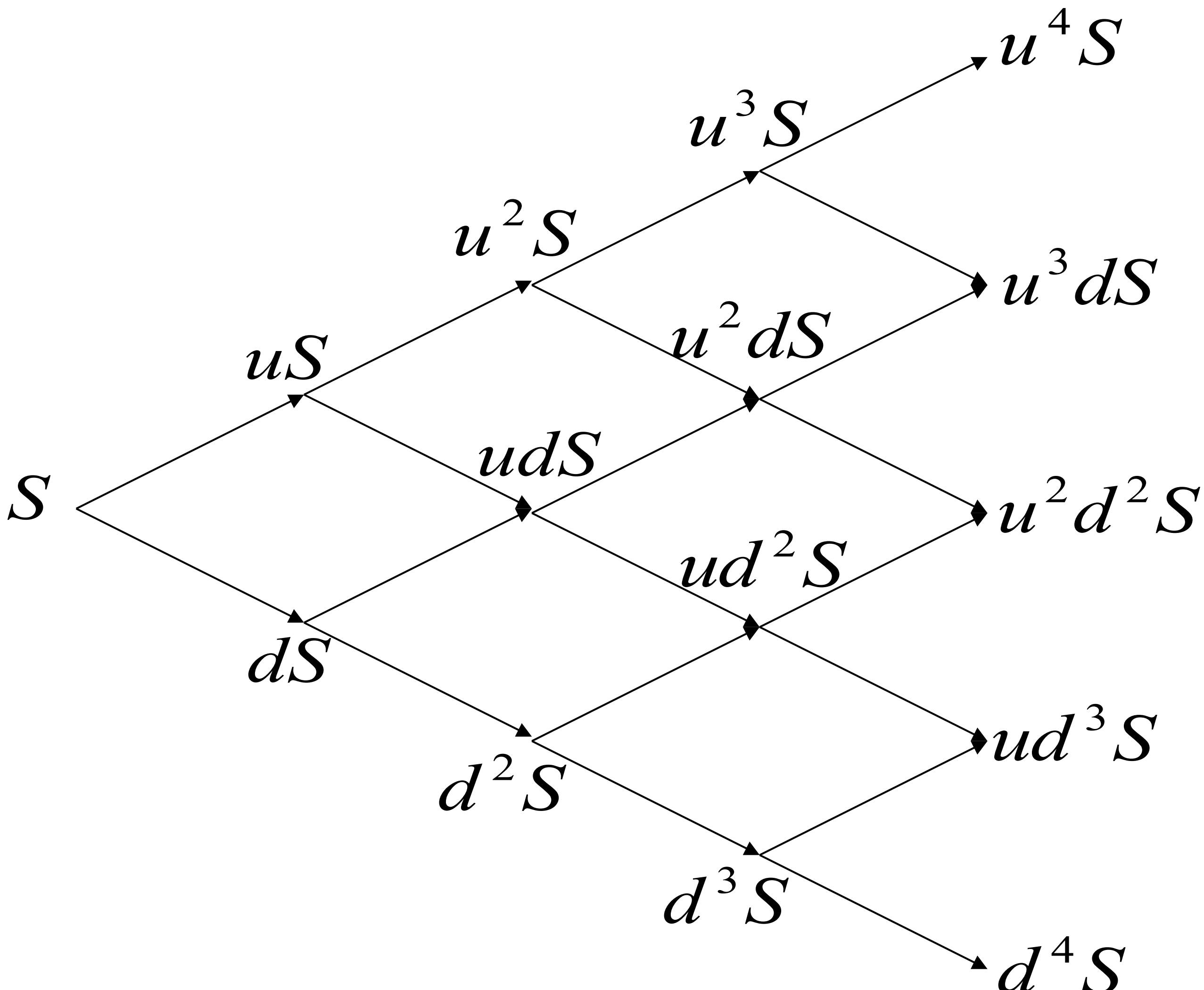
$$c + Ke^{-rT} = p + S.$$

- This equation represents the put-call parity relationship, aka the "Fundamental Theorem of Financial Engineering".
- Note that once we know the price of 3 out of 4 component securities, then we can find the price of the 4th security. So in the previous numerical example, once we calculated $c =$ \$2.70, then $p = c + Ke^{-rT} - S =$ \$2.70 + \$99.58 - 100 = \$2.28.

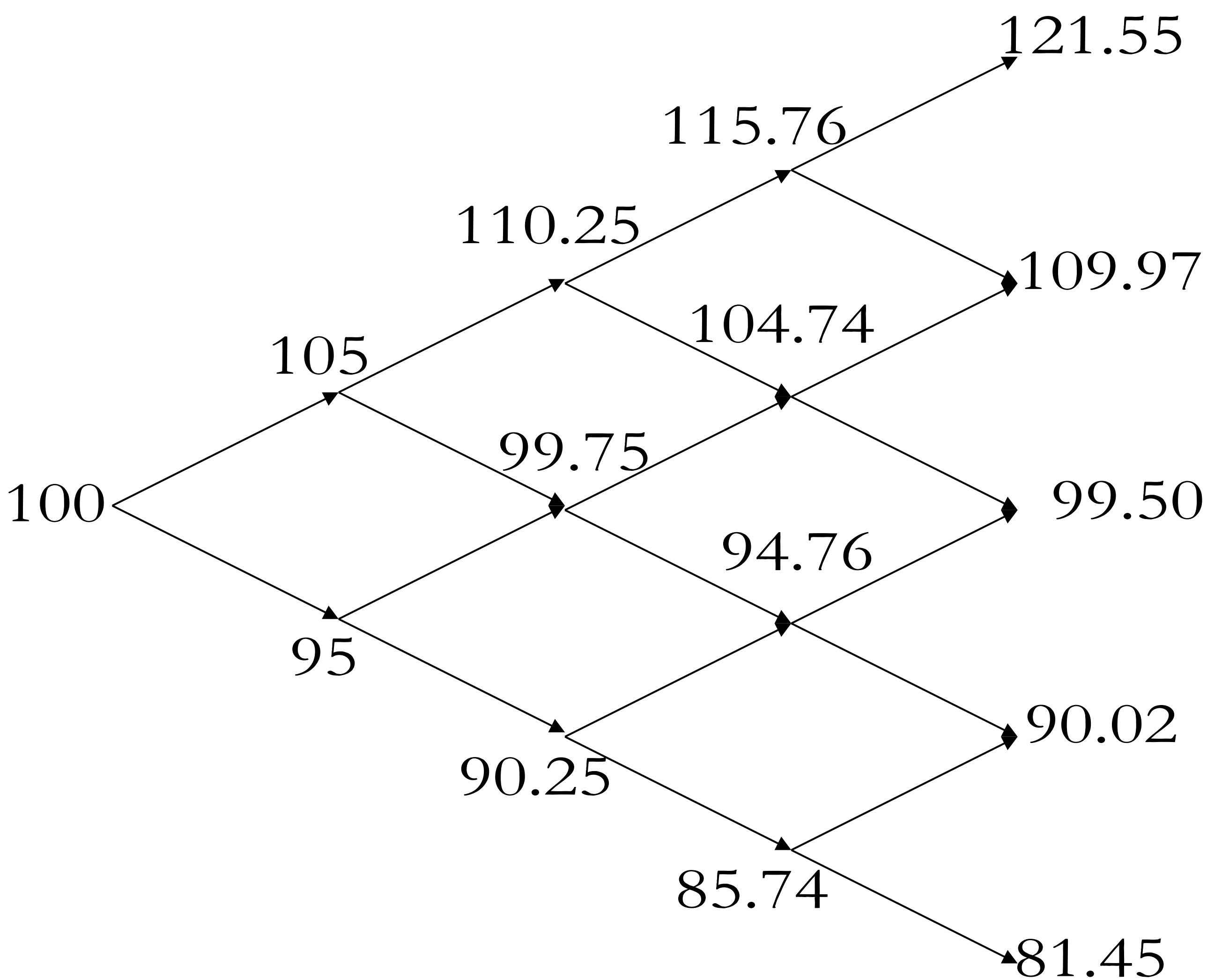
Agenda for Today

- Expanding the "binomial tree" to multiple timesteps.
- Convergence of the multi-timestep binomial formula to the Black-Scholes-Merton formula
- Coming up: Application of Black-Scholes-Merton to credit risk

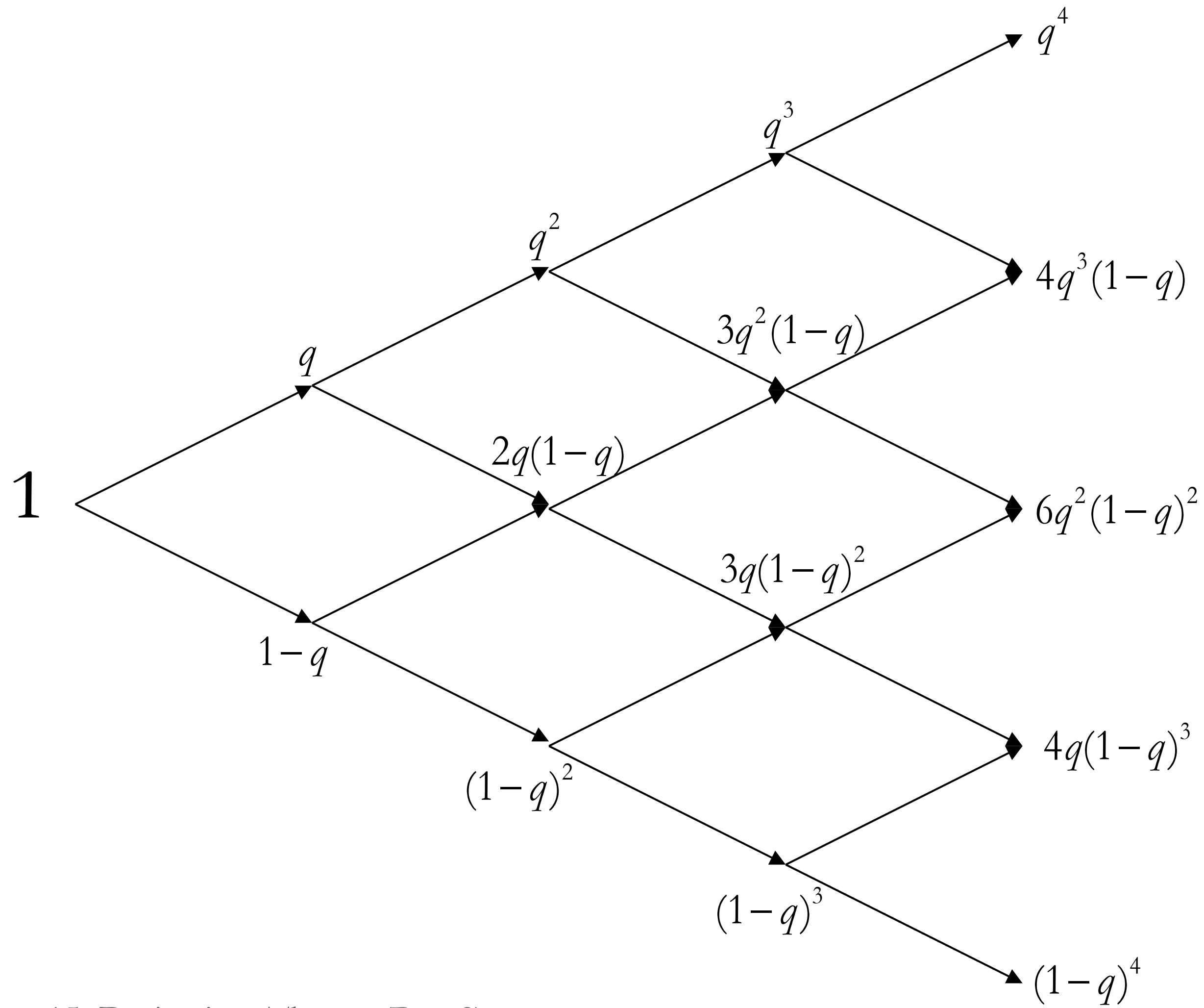
Implications of even more Time Steps



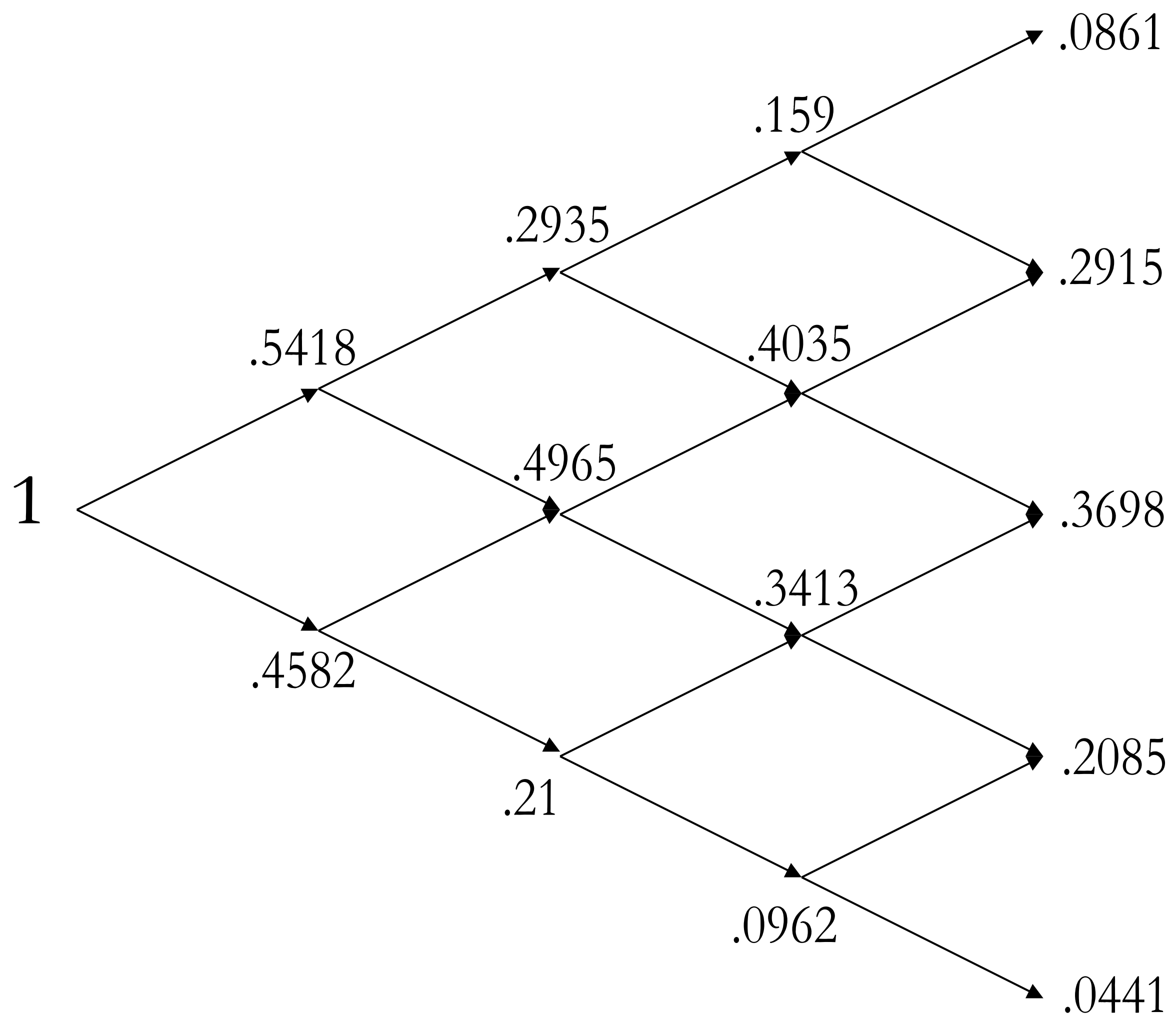
Implications of even more Time Steps



Implications of even more Time Steps



Implications of even more Time Steps



1-4 Time Step Call Option Prices

1. n = 1: By inspection, the call option is only in-the-money at the up (u) node. For one timestep,

$$c = e^{-r\delta t}[qc_u] = 0.9958(0.5418)(5) = \$2.70.$$

2. n = 2: By inspection, the call option is only in-the-money at the up-up (uu) node. For two timesteps,

$$c = e^{-2r\delta t}[q^2 c_{uu}] = 0.9958^2[.5418^2(10.25)] = \$2.98.$$

3. n = 3: By inspection, the call option is only in-the-money at the up-up-up (uuu) node and the up-up-down (uud) node. For three timesteps,

$$c = e^{-3r\delta t}[q^3 c_{uuu} + 3q^2(1-q)c_{uud}]$$
$$= 0.9958^3\left[(0.5418^3)(15.76) + 3(0.5418^2)(0.4582)(4.74)\right] = \$4.36.$$

4. n = 4: By inspection, the call option is only in-the-money at the up-up-up-up ($uuuu$) and up-up-up-down ($uuud$) node. For four timesteps,

$$c = e^{-4r\delta t}[q^4 c_{uuuu} + 4q^3(1-q)c_{uuud}]$$
$$= 0.9958^4\left[(0.5418^4)(21.55) + 4(0.5418^3)(0.4582)(9.97)\right] = \$4.68.$$

1-4 Time Step Put Option Prices

Having identified arbitrage-free prices for 1-4 time step call options, we can apply the put-call parity equation to determine arbitrage-free prices for otherwise identical (same underlying asset, exercise price, and time to expiration) 1-4 time step put options:

$$c + Ke^{-rn\delta t} = p + S.$$

The Cox-Ross-Rubinstein (CRR) call equation

- The complexity of analysis grows with each additional time-step. Fortunately, Cox, Ross, and Rubinstein (CRR) provide a recursive multi-period call option pricing formula:

$$C = e^{-rT} \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} q^j (1-q)^{n-j} C_j .$$

- $\frac{n!}{j!(n-j)!}$ indicates how many path sequences exist for each of the $n+1$ terminal nodes;
- $q^j (1-q)^{n-j}$ corresponds to the risk-neutral probability of one j up and $n-j$ down move path sequence;
- $\frac{n!}{j!(n-j)!} q^j (1-q)^{n-j}$ indicates the risk-neutral probability of the stock price ending up at the $j, n-j$ terminal node;
- $C_j = Max\left[0, u^j d^{n-j} S - K\right]$; and
- $T = n\delta t$ corresponds to a fixed expiration date T periods from now.

The Cox-Ross-Rubinstein (CRR) call equation

- Since $C_j = Max\left(0, u^j d^{n-j} S - K\right)$, we need to determine the minimum number of up moves such that the call option expires in-the-money; i.e., so that $u^j d^{n-j} S > K$.
- Let b represent the *non-integer* value for j such that $u^b d^{n-b} S = K$. Solving this equation for b,

$$\ln\left(u^b d^{n-b} S\right) = \ln K$$

$$b \ln u + (n-b) \ln d = \ln\left(K / S\right);$$

$$b \ln\left(u / d\right) = \ln\left(K / Sd^n\right);$$

$$b = \ln\left(K / Sd^n\right) / \ln\left(u / d\right).$$

- The minimum *integer* value for j is a, obtained by rounding to the nearest integer greater than b.
- If $a = 0$, then the call is in-the-money at *all* $n+1$ terminal nodes.
- If $a = n+1$, the call is out-of-the-money at *all* $n+1$ terminal nodes.

The Cox-Ross-Rubinstein (CRR) call equation

- Having determined the *minimum* number of up moves (a) required for $C_j > 0$, if follows that $C_j > 0$ for $j = a,\ldots,n$. Then the risk neutral valuation formula for pricing such an option is:

$$C = SB_1 - Ke^{-rT}B_2,$$

where

$$B_1 = \sum_{j=a}^{n}\left(\frac{n!}{j!(n-j)!}\right)\cdot q^j \cdot (1-q)^{n-j} \cdot \left(u^j \cdot d^{n-j} \cdot e^{-rn\delta t}\right);\text{ and}$$

$$B_2 = \sum_{j=a}^{n}\left(\frac{n!}{j!(n-j)!}\right)\cdot q^j \cdot (1-q)^{n-j}.$$

The Black-Scholes-Merton (BSM) call equation

- As in the previous slide, suppose the time to expiration $T = n\delta t$. Now consider the "limiting" case where $n \to \infty$ and $\delta t \to 0$ for a fixed value of T. When this occurs, the binomial risk neutral probabilities B_1 and B_2 that appear in the *CRR* option pricing formula converge in probability to the standard normal probabilities $N(d_1)$ and $N(d_2)$, where
  $$d_1 = \frac{\ln(S/K) + (r + .5\sigma^2)T}{\sigma\sqrt{T}} \text{ and } d_2 = d_1 - \sigma\sqrt{T}.$$
- Then the risk neutral valuation formula for pricing such an option is:
  $$C = SN(d_1) - Ke^{-rT}N(d_2).$$
  This formula was independently published by Black and Scholes and by Merton in 1973, so it is commonly referred to as the BSM call option pricing formula. Scholes and Merton were awarded the Nobel Prize for Economics in 1997 for this discovery; Black was not named since he passed away in 1995 and the Nobel Prize is not posthumously given.

The CRR and BSM put equations

- The CRR and BSM put equations are obtained by invoking the put-call parity theorem. Since the only difference between these equations is that CRR is based upon the standard binomial distribution function (as captured by B_1 and B_2) whereas BSM is based upon the normal distribution function (as captured by $N(d_1)$ and $N(d_2)$), the CRR and BSM put equations are otherwise identical to each other.
- According to the put call parity theorem, the BSM put equation is

$$\begin{aligned} p &= c + Ke^{-rT} - S \\ &= SN(d_1) - Ke^{-rT}N(d_2) + Ke^{-rT} - S \\ &= Ke^{-rT}(1 - N(d_2)) - S(1 - N(d_1)). \end{aligned}$$

- By symmetry, it follows that the CRR put equation is

$$p = Ke^{-rT}(1 - B_2) - S(1 - B_1).$$

CRR & BSM call and put prices – numerical example

Suppose $S = \$100$, $K = \$100$, $\sigma = .20$, $n = 2$, $\delta t = .25$, $T = n\delta t = 2(.25) = .5$, and $r = .03$. Also suppose that $u = e^{\sigma\sqrt{\delta t}} = e^{.2\sqrt{.25}} = 1.1052$, $d = 1/u = .9048$, and $q = \dfrac{e^{r\delta t} - d}{u - d} = .5126$. What are the CRR call and put prices, given these parameters?

SOLUTION: Here's the two-timestep stock tree:

		\$122.14
	\$110.52	
\$100.00		\$100.00
	\$90.48	
		\$81.87

Therefore, the only node at which this call is in-the-money is a node *uu*; specifically, $c_{uu} = \max[0, S_{uu} - K] = \$22.14$ and $c_{ud} = c_{dd} = 0$. Then

$$c = e^{-rT}[q^2 c_{uu}] = .9851(.5126^2 \cdot \$22.14) = \$5.73, \text{ and}$$

$$p = c + e^{-rT}K - S = \$5.73 + \$98.51 - \$100 = \$4.24.$$

CRR & BSM call and put prices – numerical example

What are the BSM call and put prices, given the parameters from the preceding page?

SOLUTION: First calculate the standard normal probabilities $N(d_1)$ and $N(d_2)$.

$$d_1 = \frac{\ln(S/K) + (r + .5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(100/100) + (.03 + .5(.04)).5}{.2\sqrt{.5}} = .1768, \text{ and}$$

$$d_2 = d_1 - \sigma\sqrt{T} = .1768 - .2\sqrt{.5} = .0354.$$

Thus, $N(d_1) = .5702$ and $N(d_2) = .5141$, and

$$c = SN(d_1) - Ke^{-rT}N(d_2) = 100(.5702) - 100e^{-.03(.5)}(.5141) = \$6.37, \text{ and}$$

$$p = Ke^{-rT}(1 - N(d_2)) - S(1 - N(d_1)) = 100e^{-.03(.5)}(.4859) - 100(.4298) = \$4.88.$$

CRR & BSM call and put prices – numerical example

- When there are only two timesteps until expiration 6 months from now, the CRR model produces call and put prices of $5.73 and $4.24, compared with BSM model prices of $6.37 and $4.88.
- However since the standard binomial distribution functions (B_1 and B_2) converge in probability to the standard normal distribution functions ($N(d_1)$ and $N(d_2)$), CRR and BSM prices also converge rather quickly. Here's a table illustrating this for 1 through 5,000 timesteps occurring during the course of a 6-month time to expiration:

Timesteps	BSM Call	BSM Put	CRR Call	CRR Put	Call Difference	Call % Diff	Put Difference
1	$6.3710	$4.8822	$7.7512	$6.2624	-$1.3802	-21.66%	-$1.3802
2	$6.3710	$4.8822	$5.7309	$4.2421	$0.6401	10.05%	$0.6401
5	$6.3710	$4.8822	$6.6501	$5.1613	-$0.2791	-4.38%	-$0.2791
10	$6.3710	$4.8822	$6.2323	$4.7435	$0.1387	2.18%	$0.1387
25	$6.3710	$4.8822	$6.4261	$4.9373	-$0.0551	-0.86%	-$0.0551
50	$6.3710	$4.8822	$6.3430	$4.8542	$0.0280	0.44%	$0.0280
100	$6.3710	$4.8822	$6.3570	$4.8682	$0.0140	0.22%	$0.0140
200	$6.3710	$4.8822	$6.3640	$4.8752	$0.0070	0.11%	$0.0070
400	$6.3710	$4.8822	$6.3675	$4.8787	$0.0035	0.05%	$0.0035
5000	$6.3710	$4.8822	$6.3707	$4.8819	$0.0003	0.00%	$0.0003

Black-Scholes-Merton (BSM) is a "limiting" case of CRR!

- See the Cox-Ross-Rubinstein model compared with the Black-Scholes-Merton model spreadsheet!