Mathematics Tutorial

Half of all Americans do not understand math, and the other two-thirds don't care.

-- Garrison Keillor

Baseball is 90% mental, the other half is physical.

--Yogi Berra

(Just about) all the math you will need

- Next, we turn our attention to a study of (most of) the mathematical principles upon which this course is based.
- In this lecture. . .
  - Common logarithms (Rule of 72), the number e, and natural logarithms
  - differentiating and Taylor series

Common Logarithms (Rule of 72)

- This a method (using common (base 10) logarithms) for estimating how long it takes for a sum to double.
- Note that $FV = (1+r)^t PV$. Suppose $FV=2PV$ and $r = 10\%$. Then $2 = (1.1)^t$.
- Solving for t,

$$t = \log_{1.1} 2 = \frac{\log_{10} 2}{\log_{10} 1.1} = \frac{0.30103}{.041393} = 7.273.$$

- Note that $r \times t = 10 \times 7.273 = 72.73$; thus $t \cong 72 / r$.

The Number e

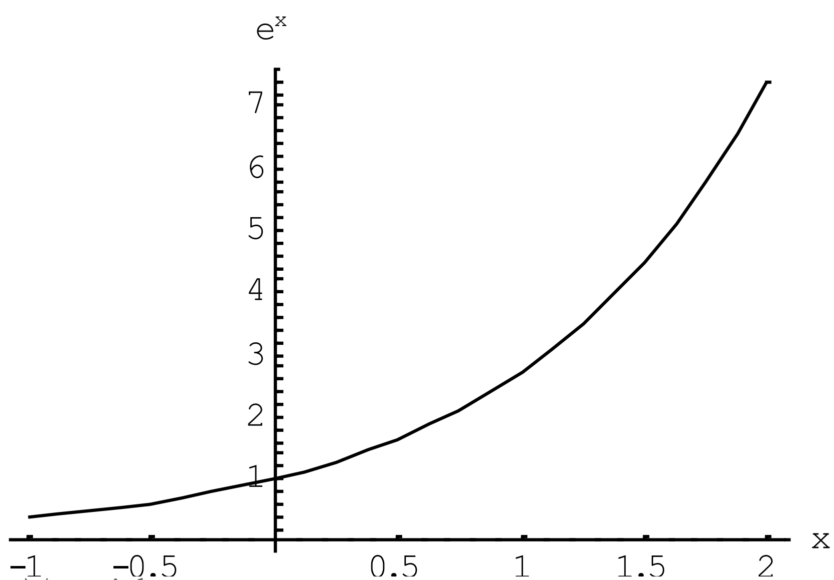
- a real (irrational) number, equal to 2.71828182845905…
- Suppose we have a function $y = f(x) = e^x$; we may also write this function $y = exp(x)$.
- The function $y = e^x = 2.71828182845905\ldots^x$; $y = e^2 = 7.38905609893065\ldots$, $y = e^1 = 2.71828182845905\ldots$, and $e^0 = 1$.

The function $y = e^x$

- The function $y = e^x$ is the solution to the following infinite series:

$$e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \ldots = \sum_{i=0}^{\infty} \frac{x^i}{i!}.$$

- Graphically, $y = e^x$ looks like this:



The function $y = e^x$

- The function $y = e^x$ has the special property that the slope, or gradient of the function is also e^x.
- Plot this slope as a function of x and you will obtain the same curve again.
- In fact, the slope of the slope of the slope....of the slope of $y = e^x$ is also e^x.

Time Value Application involving e

The present value of $\$X$ received T periods from today is $PV(\$X) = \$X/(1+r)^T$. Suppose that compounding occurs m times per year. With more frequent compounding, the present value of $1 will be lower; specifically, $PV(\$X) = \$X/(1+\frac{r}{m})^{mT}$. With compounding occurring over infinitesimally small periods of time, the present value of $\$X$ equals $\underbrace{lim}_{m\to\infty} PV(\$X) = \$Xe^{-rT}$.

Similarly, the future value as of time T of $\$X$ received at time 0 is $FV(\$X) = \$X(1+r)^T$; with more frequent compounding, the future value will be higher; specifically, $FV(\$X) = \$X(1+r/m)^{mT}$. With compounding occurring over infinitesimally small periods of time, the future value of $1 equals $\underbrace{lim}_{m\to\infty} FV(\$X) = \$Xe^{rT}$.

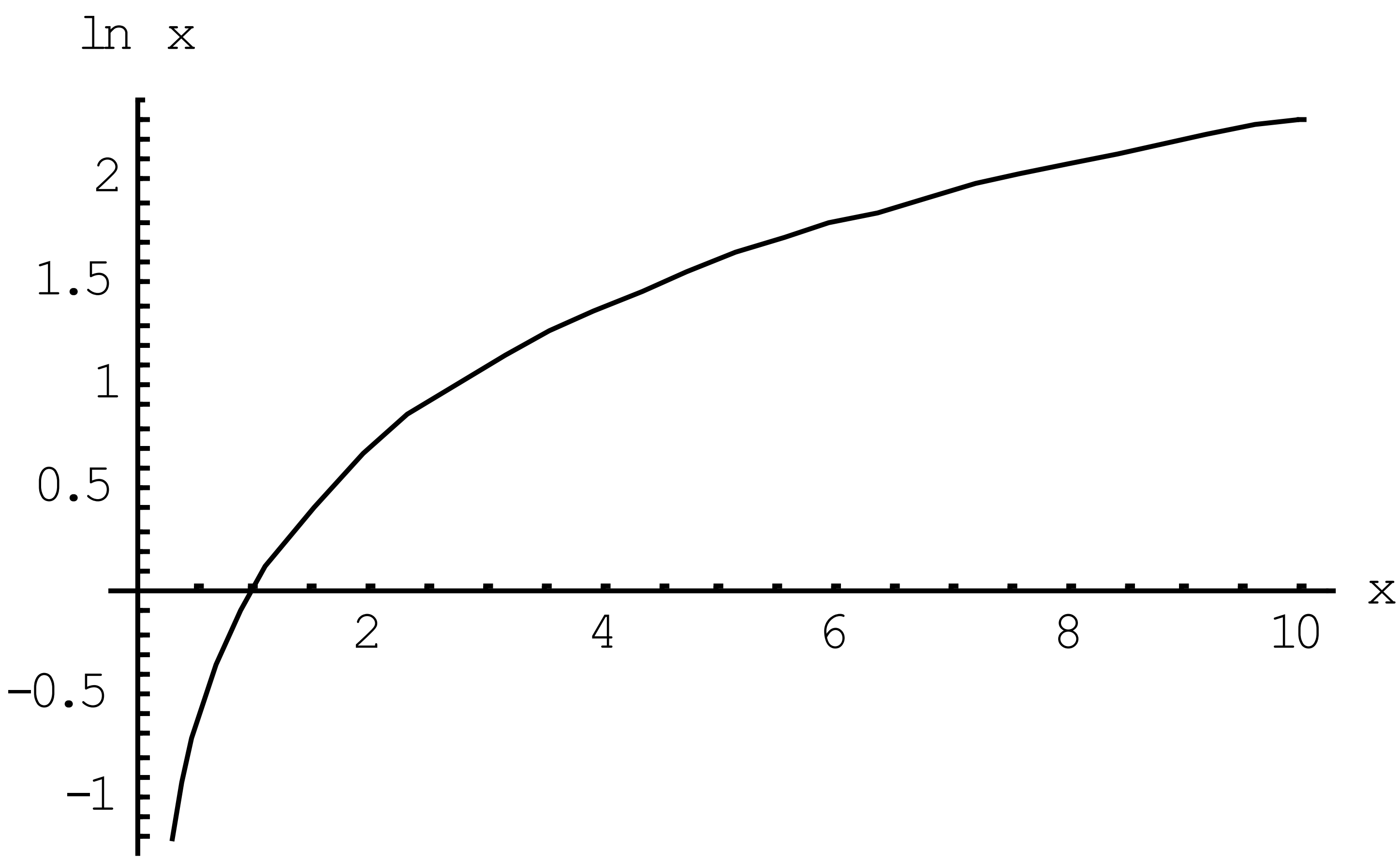
Time Value Application involving e

- Present value of \$100 received 10 years from today:
  - 5% interest rate with annual compounding: $\$100/1.05^{10} = \$100(.6139) = \$61.39$.
  - 5% interest rate with continuous compounding: $\$100e^{-.05(10)} = \$100(0.6065) = \$60.65$.

The natural logarithm (*ln*) function

- Take the plot of e^x and rotate it about a 45° line. This new function is $ln\ x$, the Napierian, or "natural" logarithm of x.
- The relationship between *ln* and *e* is $e^{\ln x} = x$ or $ln(e^x) = x$; consequently, they are inverses of each other.
- The slope of the $ln\ x$ function is x^{-1}.

The natural logarithm (*ln*) function



Calculating Derivatives

- The definition of the derivative of the function $y = f(x)$ with respect to x is:

$$f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Example 1 (line): Suppose $y = 10+5x$. Then

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \left[\frac{(10+5x+5\Delta x) - (10+5x)}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{5\Delta x}{\Delta x} \right] = 5.$$

Calculating Derivatives

- **Example 2 (parabola):** Suppose $y = x^2$. Then

$$f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0}\left[\frac{(x+\Delta x)^2 - x^2}{\Delta x}\right] = 2x.$$

The rate of change of the parabola depends upon the particular value of x; e.g., if this derivative is evaluated at $x = 0$, then $f'(0) = 2(0) = 0$, if it is evaluated at $x = 2$, then $f'(2) = 2(2) = 4$, and if it is evaluated at $x = 4$, then $f'(4) = 2(4) = 8$.

Calculating Derivatives

- Suppose $z = f(x, y)$. Then the *partial derivative* of the function, f, with respect to x (denoted by $\partial f / \partial x$) is:

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}\right]$$

Here, $\partial f / \partial x$ is simply the ordinary derivative of f with respect to x while holding y constant.

Example 4 (multivariable function): Suppose $z = f(x, y) = 2x^2 - 3x^2y + 5y + 1$. Then $\partial f / \partial x = 4x - 6xy$ and $\partial f / \partial y = -3x^2 + 5$.

Optimization

- Optimization (e.g., maximization or minimization) requires basic calculus.
- For example, suppose your firm produces only one product, and you are interested in determining the profit maximizing number of units (Q) to produce. The price per unit is \$30. Your fixed costs are \$40, and your variable costs are $\$3Q^2$. Thus, your profit equation is:

$$\pi = \underbrace{30Q}_{\text{total revenue}} \underbrace{-40}_{\text{fixed cost}} - \underbrace{3Q^2}_{\text{variable cost}} .$$

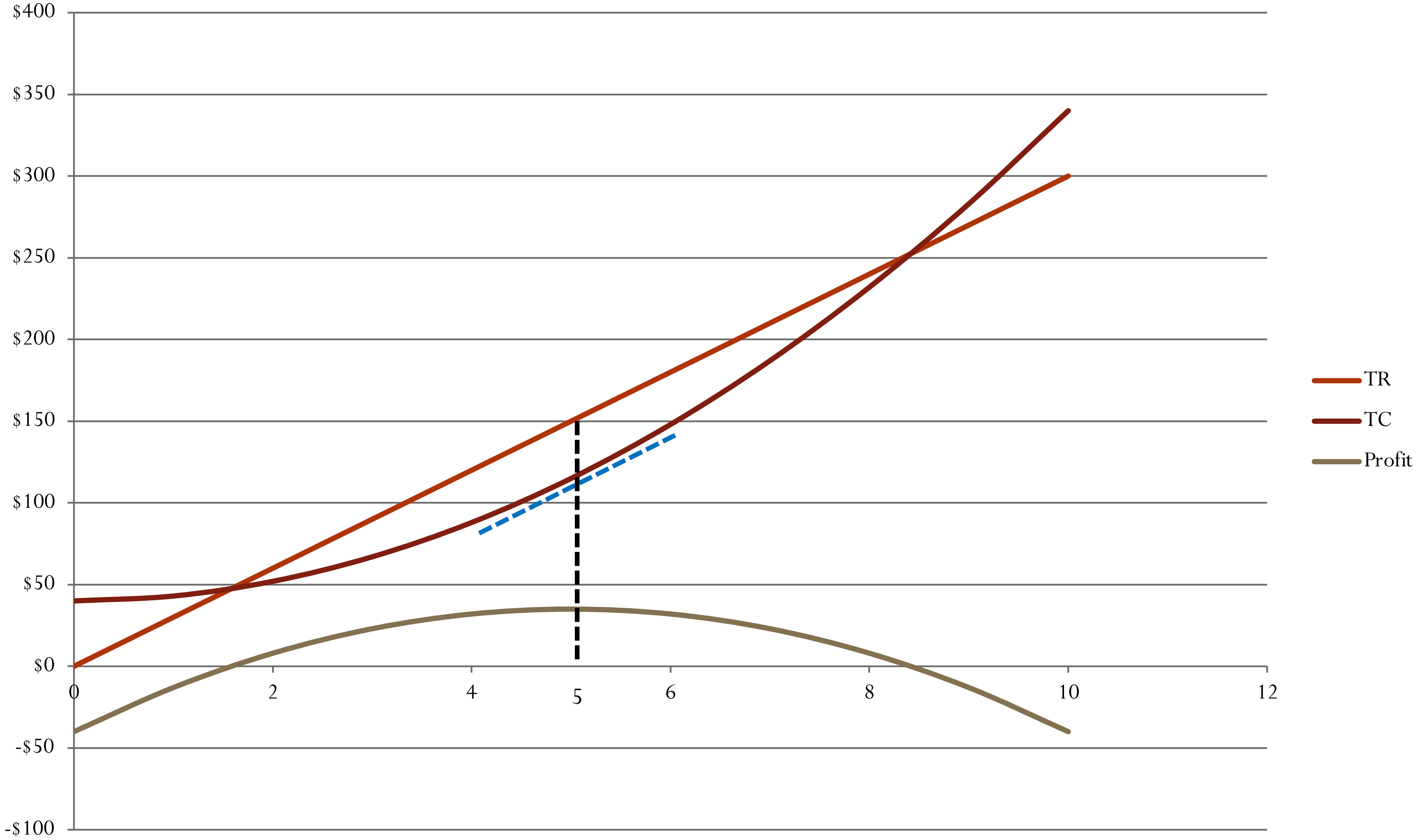
Optimization

- Next, we'll do the math. Since the slope of the profit function is zero at its maximum, we calculate the first derivative of profit (π) with respect to quantity (Q), set this derivative equal to zero, and then solve for Q^*:

$$\frac{d\pi}{dQ} = 30 - 6Q^* = 0 \Rightarrow Q^* = 5.$$

This is called the *first order condition*; it is a *necessary* condition for a maximum or a minimum. In order to determine whether $Q^* = 5$ minimizes or maximizes π, we must determine whether the second derivative of profit (π) with respect to quantity (Q) is positive or negative; since $\frac{d^2\pi}{dQ^2} = -6 < 0$, $Q^* = 5$ maximizes π.

Optimization



Optimization

- When profit is maximized, the slope of the total cost curve (marginal cost, or *MC*) is equal to the slope of the total revenue line (marginal revenue, or *MR*). Here,

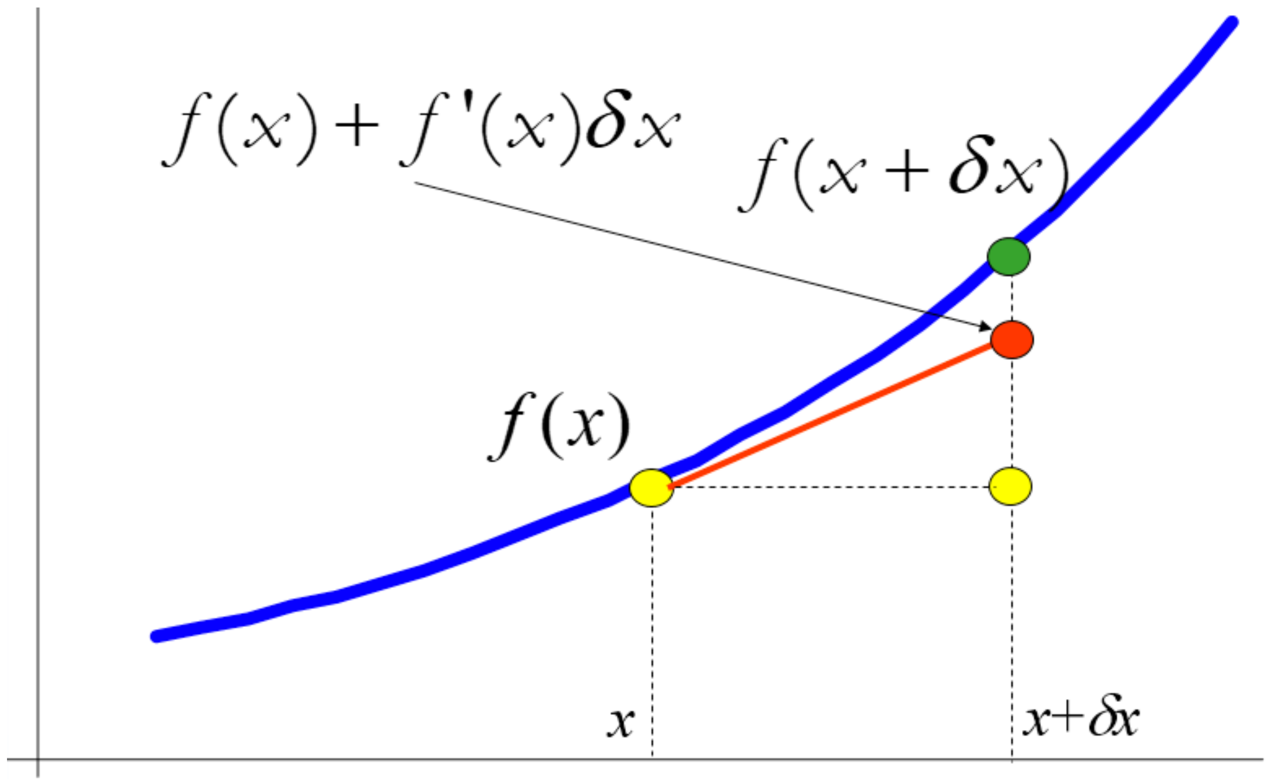
$$MR = \frac{dTR}{dQ} = P = \$30, \text{ and}$$

$$MC = \frac{dTC}{dQ} = 6Q.$$

Setting $MR = MC$, we find that $\$30 = 6Q \Rightarrow Q^* = 5$!

Using Derivatives to Approximate Functions

- A function $f(x)$ is given by the blue curve below. The value of the y coordinate $f(x+\delta x)$, given $f(x)$, can be approximated using Taylor polynomials.
- The red line segment represents a first-order (linear) approximation ($f(x) + f'(x)\delta x$), but we can do better with higher order Taylor polynomials!



Using Derivatives to Approximate Functions

- Consider a very small (but non-zero) δx.
- The equation for the linear approximation of $f(x+\delta x)$ is $f(x+\delta x) \approx f(x) + f'(x)\delta x$.
- $f(x+\delta x) \approx f(x) + f'(x)\delta x + .5f''(x)\delta x^2$ (a quadratic approximation) gets us even closer.
- The n^{th} order approximation of $f(x+\delta x)$ is written

$$f(x+\delta x) \approx f(x) + \sum_{i=1}^{n} \frac{1}{i!} \delta x^{i} \frac{d^{i} f(x)}{dx^{i}}.$$

- This is commonly referred to as an n^{th} order Taylor series.

Numerical Example of Taylor polynomials

- Next, we compute the first four Taylor polynomials of $f(x) = e^x$ at x=0 and sketch their graphs.
  - Since we are interested in expansions around x=0, we must evaluate the first four derivatives of $f(x)$ evaluated at x=0.
  - Note that since $f(x) = e^x$, $f'(x)=f''(x)=f'''(x)= f''''(x)= e^x$, and $f'(0)= f''(0)=f'''(0) = f''''(0) = 1$!

Numerical Example of Taylor polynomials

- 1^{st} Order Taylor Polynomial:

$$f(0+\delta x) \approx f(0) + f'(0)\delta x = 1 + \delta x;$$

- 2^{nd} Order Taylor Polynomial:

$$f(0+\delta x) \approx f(0) + f'(0)\delta x + (1/2)f''(0)\delta x^2$$
$$= 1 + \delta x + (1/2)\delta x^2;$$

- 3^{rd} Order Taylor Polynomial:

$$f(0+\delta x) \approx f(0) + f'(0)\delta x + (1/2)f''(0)\delta x^2 + (1/6)f'''(0)\delta x^3$$
$$=1 + \delta x + (1/2)\delta x^2 + (1/6)\delta x^3;$$

- 4^{th} Order Taylor Polynomial:

$$f(0+\delta x) \approx f(0) + f'(0)\delta x + (1/2)f''(0)\delta x^2$$
$$+ (1/6)f'''(0)\delta x^3 + (1/24)f''''(0)\delta x^4$$
$$=1 + \delta x + (1/2)\delta x^2 + (1/6)\delta x^3 + (1/24)\delta x^4.$$

Numerical Example of Taylor polynomials

δw	$e(0+\delta w)$	0rs Nqc dq	1nc Nqc dq	2qc Nqc dq	3sg Nqc dq
-2.0	0.135	-1.000	1.000	-0.333	0.333
-1.6	0.202	-0.600	0.680	-0.003	0.270
-1.2	0.301	-0.200	0.520	0.232	0.318
-0.8	0.449	0.200	0.520	0.435	0.452
-0.4	0.670	0.600	0.680	0.669	0.670
0.0	1.000	1.000	1.000	1.000	1.000
0.4	1.492	1.400	1.480	1.491	1.492
0.8	2.226	1.800	2.120	2.205	2.222
1.2	3.320	2.200	2.920	3.208	3.294
1.6	4.953	2.600	3.880	4.563	4.836
2.0	7.389	3.000	5.000	6.333	7.000

Comparison of Taylor Polynomials for *exp*(*x*=0)