Portfolio and Capital Market Theory

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Abstract

The foundations of portfolio and capital market theory were set forth in seminal articles published during the 1950's and 1960's by Markowitz (1952), Tobin (1958), Sharpe (1964), Lintner (1965), and Mossin (1966). The purpose of this teaching note is to provide a brief and succinct synthesis of the contributions made by these and other articles in the development of portfolio and capital market theory.

1 Introduction

Portfolio theory involves the study of how risk averse investors can construct portfolios in order to optimize the tradeoff between risk (as measured by variance) and expected return. The theory emphasizes the necessity of analyzing risk in a *portfolio* context. After all, the total risk of a portfolio depends not only on the unique risks of the securities which comprise the portfolio, but also on the ways these risks interact with each other. Capital market theory addresses the implications of portfolio theory for the pricing of risk in the capital markets.

The statistical foundations for portfolio and capital market theory are based on the Central Limit Theorem and the Law of Large Numbers. According to the Central Limit Theorem, as individual probability distributions are aggregated, the average distribution converges in probability toward the normal distribution (assuming that the means and variances of the individual probability distributions exist).[1,2] Furthermore, if portfolio return

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[1]Technically, the "classical" version of the Central Limit Theorem requires that random variables be independent and identically distributed random variables. However, Godwin and Zaremba (1961) note that it is "well-known" that the Central Limit Theorem can be extended to cases in which the random variables under consideration are not entirely independent, as is the case here.

[2]A famous example of where the mean and variance of the individual probability distributions *do not* exist is the St. Petersburg Paradox (see Bernoulli (1954)). Bernoulli proposes the following gamble: "Peter tosses a coin and continues to do so until it should land "heads" when it comes to the ground. He agrees to give Paul one ducat if he gets "heads" on the very first throw, two ducats if he gets it on the second, four if on the third, eight if on the fourth, and so on, so that with each additional throw the number of

distributions are normally distributed, then this implies that variance is a "complete" risk measure. Therefore, arbitrarily risk averse investors need only consider expected value and variance in order to maximize expected utility.[3]

The Law of Large Numbers plays a particularly important role in capital market theory. To see this, consider first the equation for portfolio variance. This parameter is obtained by calculating a weighted average of all of the variances and covariances of the portfolio's component securities, as shown by equation (1):

$$\sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij}, \tag{1}$$

where w_i is the proportion of total investment in security i, σ_i^2 is the variance of security i, and $\sigma_{i,j}$ is the covariance between securities i and j. Now suppose that risks are identically distributed; i.e., all securities have the same expected return, the same variance, and the same covariances. By imposing these restrictive assumptions, the risk of an n security portfolio simplifies from equation (1) to equation (2):

$$\sigma_p^2 = \frac{\sigma^2}{n} + \frac{n-1}{n}\rho\sigma^2, \tag{2}$$

where ρ represents the correlation coefficient held in common by all pairwise combinations of the n securities. Equation (2) provides us with the familiar characterization of portfolio risk as consisting of two components: unsystematic risk ($\frac{\sigma^2}{n}$) and systematic risk ($\frac{n-1}{n}\rho\sigma^2$). In the limit, as the number of risks becomes arbitrarily large (i.e., as $n \rightarrow \infty$), then $\sigma_p^2 = \rho\sigma^2$. In other words, there is only covariance risk. Since investors are risk averse, they will fully diversify their portfolios so that the only source of risk that remains (and is priced) is the covariance risk that is inherent in the economy.

2 Portfolio Theory

2.1 Portfolio expected return and risk calculations

We calculate expected returns, standard deviations, and covariances on individual securities as follows:

$$E(r_i) = \sum_{s=1}^{n} p_s r_{i,s}, \tag{3}$$

ducats he must pay is doubled." The probability that it will take n coin tosses in order for heads to come up is $.5^n$, and the payoff after the n^{th} coin toss is 2^{n-1} ducats; thus the expected value of this game is $EV = \sum_{i=1}^{\infty} .5^i 2^{i-1} = \sum_{i=1}^{\infty} .5 \Rightarrow \infty$. The "paradox" is that the value of this gamble cannot possibly be its expected monetary payoff, since as Bernoulli notes, "... no one would be willing to purchase it (this gamble) at a moderately high price.

[3]The notion that variance is a "complete" risk measure when return distributions are normally distributed was first shown by Ross (1978). Under the normal distribution, there are only two parameters, mean (expected value) μ and variance σ^2. Since the risk of a normally distributed random variable is fully captured by variance, it follows (see equation (A.2) in the appendix) that expected utility for a normally distributed risk depends only on the expected value and the variance of such a risk.

$$\sigma_i = \sqrt{\sum_{s=1}^{n} p_s (r_{i,s} - E(r_i))^2}, \text{ and} \tag{4}$$

$$\sigma_{i,j} = \sum_{s=1}^{n} p_s (r_{i,s} - E(r_i))(r_{j,s} - E(r_j)), \tag{5}$$

where p_s is the probability that state s will occur, $r_{i,s}$ represents the state-contingent return on the i^{th} security. Portfolio expected returns and standard deviations are calculated as follows:

$$E(r_p) = \sum_{i=1}^{n} w_i E(r_i), \text{ and} \tag{6}$$

$$\sigma_p = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{i,j}}. \tag{7}$$

2.2 Mean-variance efficiency

Our next task is to determine the subset of portfolios that satisfy the mean-variance efficiency criterion. A portfolio is said to be mean-variance efficient if there is no other portfolio which, for a given level of expected return, has lower risk (as measured by variance). Equivalently, a mean-variance efficient portfolio has the property of having the highest expected return for a given level of risk. Markowitz (1952) invented a mathematical programming technique for finding the "efficient frontier" for which he won the Nobel Prize in 1990.[4]

The basic mathematical program proposed by Markowitz is as follows:

$$\underset{\{w_1, w_2, \ldots, w_n\}}{\text{Minimize}} \ \sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}, \tag{8}$$

subject to $E(r_p) = \sum_{i=1}^{n} w_i E(r_i) = \chi$ and $\sum_{i=1}^{n} w_i = 1$. Equivalently, one may solve the following mathematical program:

$$\underset{\{w_1, w_2, \ldots, w_n\}}{\text{Maximize}} \ E(r_p) = \sum_{i=1}^{n} w_i E(r_i), \tag{9}$$

subject to $\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij} = \delta$ and $\sum_{i=1}^{n} w_i = 1$. Mathematical programming theory tells us that the constrained optimization problems represented by equations (8) and (9) represent

[4] The efficient frontier corresponds to that subset of portfolios which satisfy the mean-variance efficiency criterion cited here. When graphed in $\{E(r_p), \sigma_p\}$ space, this corresponds to the northwest perimeter of the "feasible" set of portfolios, which consists of all possible portfolio combinations of risky securities, including mean-variance efficient portfolios. Portfolios which lie below the efficient frontier are mean-variance inefficient, since they lack adequate expected return, given the level of risk. Similarly, portfolios that cluster to the right of the efficient frontier are mean-variance inefficient because risk is too high for the level of expected return.

the "prime" and "dual" programs for the determination of mean-variance efficient portfolios, and consequently the 1 x n vector of security weights $(w_1, w_2, ..., w_n)$ determined by either method will be the same.

The solution procedure for equation (8) (equation (9)) involves finding the vector of security weights $(w_1, w_2, ..., w_n)$ for a *given* value of χ (δ), and then iteratively solving for other $(w_1, w_2, ..., w_n)$ vectors involving higher values for χ (δ). A special case is the minimum variance portfolio, which is the end point of the efficient frontier. For simplicity (without loss of generality), consider the determination of the minimum variance portfolio when there are only 2 risky securities. For a two-asset portfolio, portfolio variance is written:

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12}. \tag{10}$$

Since the security weights sum to 1; i.e., $w_1 + w_2 = 1$, by substituting $1 - w_1$ in place of w_2 on the right hand side of equation (10), the equation for portfolio variance σ_p^2 now has only one unknown, which is w_1:

$$\begin{aligned}\sigma_p^2 &= w_1^2\sigma_1^2 + (1-w_1)^2\sigma_2^2 + 2w_1(1-w_1)\sigma_{12} \\ &= w_1^2(\sigma_1^2 + \sigma_2^2) + 2w_1(\sigma_{12} - \sigma_2^2) + \sigma_2^2 - 2w_1^2\sigma_{12}.\end{aligned} \tag{11}$$

Since we are interested in finding the least risky combination of securities 1 and 2 for which the expected return constraint is non-binding, we differentiate equation (11) with respect to w_1, set the result equal to zero, and solve for w_1. Therefore,

$$\begin{aligned}\frac{d\sigma_p^2}{dw_1} &= 2w_1(\sigma_1^2 + \sigma_2^2) + 2(\sigma_{12} - \sigma_2^2) - 4w_1\sigma_{12} \\ &= w_1(\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}) + \sigma_{12} - \sigma_2^2 = 0 \Rightarrow w_1 = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}.\end{aligned} \tag{12}$$

By weighting 2-asset portfolios according to the ratio given by equation (12), we are guaranteed a portfolio combination which *minimizes* total portfolio risk. We can determine the mean-variance efficient set of 2-asset portfolios by starting with this portfolio and iteratively solving the ($n = 2$ version of) equation (8) for higher values of χ (i.e., expected portfolio return).[5]

2.3 Mean-variance efficiency when there are n securities

Given $E(r_i)$, σ_i, and σ_{ij}, when there are n securities the investor must 1) determine which combinations of the n securities are mean-variance efficient, and 2) select a portfolio from the efficient set; this involves finding the portfolio that maximizes expected utility. Depending on the investor's degree of risk aversion, he or she may select a relatively safe or risky portfolio. However, irrespective of the degree of risk aversion, so long as investors have the same beliefs concerning risk and return for securities, then everyone selects from the same set of mean-variance efficient portfolios.

[5]Similarly, in the general n security case, we first find the least risky combination of securities 1 through n for which equation (8)'s expected return constraint is non-binding, and then iteratively solve equation (8) for other $(w_1, w_2, ..., w_n)$ vectors that result in higher values for χ.

2.4 Computation of the investor's indifference curves

Next, we turn our attention to the issue of how to select an "optimal" portfolio. As explained in the Appendix and shown in equation (A.5), the certainty-equivalent portfolio return (r_p^c) for a risk averse investor is $r_p^c = E(r_p) - .5\sigma_p^2/\tau$, where τ corresponds to the reciprocal of the investor's relative risk aversion measure. Since maximizing expected utility is equivalent to maximizing the certainty-equivalent portfolio return, we obtain an indifference curve equation by solving equation (A.5) for $E(r_p)$:

$$E(r_p) = r_p^c + .5\sigma_p^2/\tau \tag{13}$$

The investor's expected utility is constant along any given indifference curve. In equation (13), the higher the risk tolerance τ, the flatter the curve. In Figure 1, we show indifference curves for two investors who have different degrees of risk tolerance. The more risk averse investor is assumed to have $\tau = .25$, and the less risk averse investor is assumed to have $\tau = .50$. We also vary r_p^c (shown as "rcp" in Figure 1) from 6% to 10%:

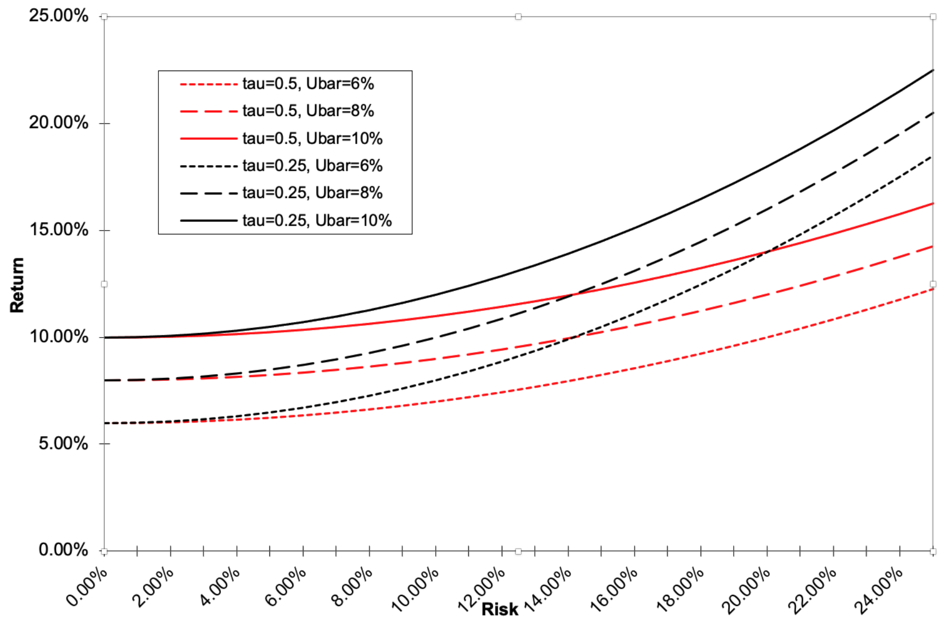

Figure 1. Indifference curves in $E(r_p), \sigma_p$ space.

2.5 Optimal portfolio selection

Now that we have defined the concepts of mean-variance efficient portfolios and indifference curves, we can make some assessments concerning actual portfolio selection. In order to maximize expected utility, the investor needs to find the point of tangency between her highest indifference curve and the efficient frontier consisting of risky securities, as illustrated by Figure 2:

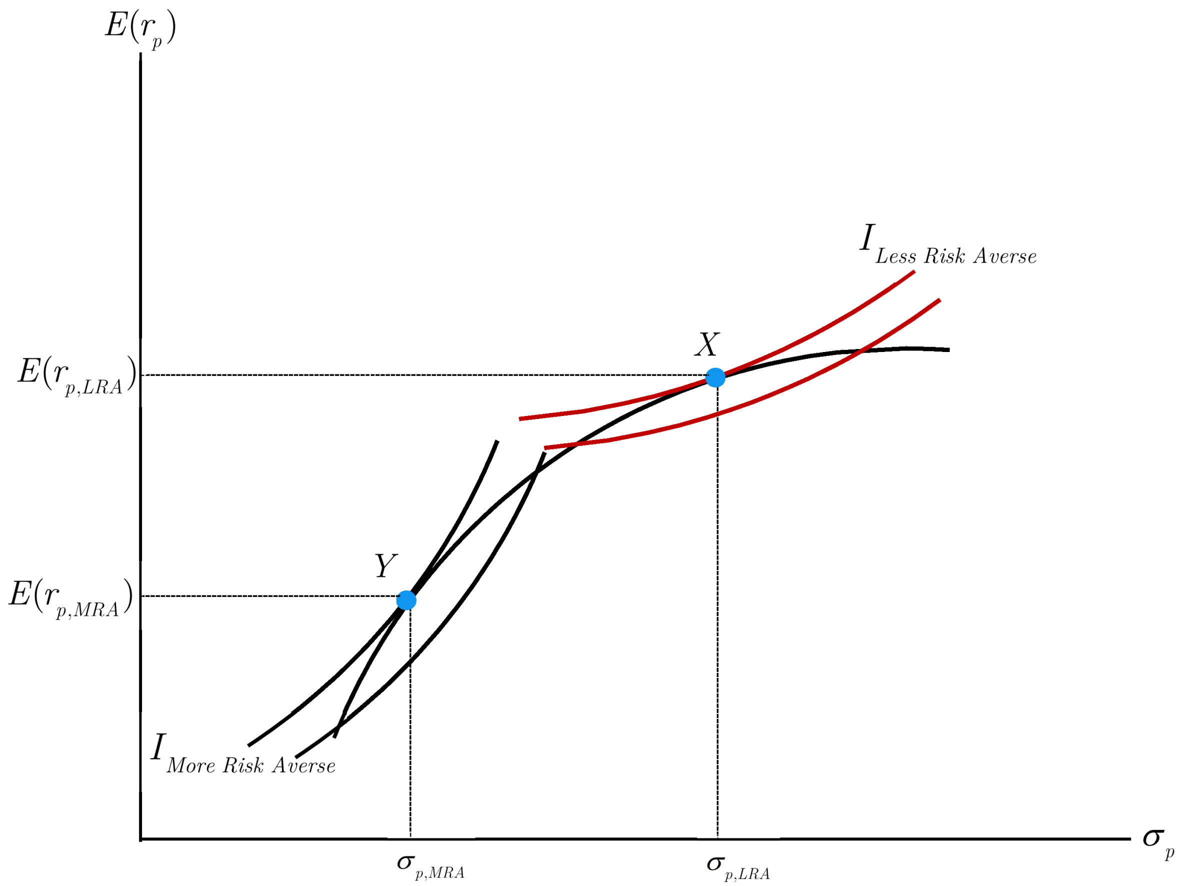

Figure 2. Optimal portfolio selection.

In Figure 2, point X corresponds to the optimal portfolio for the *less* risk averse (more risk tolerant) investor whereas point Y is the optimal portfolio for the *more* risk averse (less risk tolerant) investor.

2.6 Optimal portfolio selection when there is a riskless security

Tobin (1958) showed, among other things, that the portfolio selection problem can be simplified if we incorporate a riskless security into our analysis. Suppose the investor limits her portfolio selection to a combination of a riskless security with expected return r_f and zero variance, along with a risky security or risky asset portfolio with expected return $E(r_j)$ and variance σ_j^2. Returns on risky securities and portfolios are assumed to be normally distributed; thus, variance provides a complete measure of risk. Let α denote the proportion of the portfolio invested in the risky security. Consequently, the expected return $E(r_p)$ and variance σ_p^2 for such a portfolio are:[6]

$$E(r_p) = \alpha E(r_j) + (1 - \alpha)\, r_f, \text{and} \tag{14}$$

[6] In equation (15), the result shown there (that portfolio variance is proportional to the variance of the risky securities) follows directly from equation (10). Substituting α for w_1 and 1 - α for w_2, we obtain $\sigma_p^2 = \alpha^2\sigma_j^2 + (1-\alpha)^2\sigma_f^2 + 2\alpha(1-\alpha)\sigma_{jf}$. Since $\sigma_f^2 = \sigma_{jf} = 0$, it follows that $\sigma_p^2 = \alpha^2\sigma_j^2$.

$$\sigma_p^2 = \alpha^2 \sigma_j^2. \tag{15}$$

In this case, it is possible to obtain a simple expression for the risk-return trade-off. From equations (14) and (15), $E(r_p) = r_f + (E(r_j) - r_f)\,\alpha$ and $\sigma_p = \alpha\sigma_j$. Thus, $\alpha = \sigma_p/\sigma_j$, and by replacing α in equation 14) with the ratio σ_p/σ_j, we obtain (equation (16)):

$$E(r_p) = r_f + \frac{E(r_j) - r_f}{\sigma_j}\sigma_p. \tag{16}$$

The investor's task is to select a value for α such that expected utility is maximized. Given equation (14), our maximand is determined by substituting the right hand sides of equations (14) and (15) into equation (A.5):

$$r_p^c = \alpha E(r_j) + (1 - \alpha) r_f - (.5/\tau)\alpha^2\sigma_j^2. \tag{17}$$

Differentiating equation (17) with respect to the investor's choice variable, α, and setting the resulting expression equal to zero yields the first order condition:

$$E(r_j) - r_f - (1/\tau)\alpha\sigma_j^2 = 0. \tag{18}$$

The second order condition for a maximum, $d^2E(U(r_p))/d\alpha^2 = -(1/\tau)\sigma_j^2 < 0$, is satisfied if $\tau > 0$, as assumed.

Rearranging the first order condition given by equation (18) and solving for α results in equation (19):

$$\alpha = \frac{(E(r_j) - r_f)}{\sigma_j^2}\tau. \tag{19}$$

In equation (19), it is apparent that the proportion which will be optimally allocated to the risky security depends on two factors: 1) the excess return on the risky security per unit of variance, and 2) the investor's risk tolerance. The greater either of these are, the greater will be the investor's exposure toward the risky security.

Note that equation (19) may be rewritten as follows:

$$\alpha = \frac{(E(r_j) - r_f)}{\sigma_j}\frac{\tau}{\sigma_j}. \tag{20}$$

In equation (20), the first ratio is the well-known "Sharpe Ratio", which is a very popular "reward-to-risk" metric that is commonly used in portfolio performance measurement. By inspection, it is apparent from equation (20) that α is positively related to the Sharpe Ratio and the investor's degree of risk tolerance, and negatively related to volatility. In other words, if the risky security is expected to "outperform" (underperform) the riskless asset on a risk-adjusted basis, then the risk averse investor optimally allocates a higher (lower) proportion of her portfolio to the risky security, other things equal. On the other hand, as the Sharpe Ratio declines, the optimal allocation to the risky security also declines.

Suppose that $E(r_j) = 12\%$, $r_f = 4\%$, and $\sigma_j = 20\%$. Table 1 provides a numerical example of the effect of changes in risk tolerance on α:

Risk tolerance (τ)	α	1-α	$E(r_p)$	σ_p
1.0	200%	-100%	20%	40%
0.8	160%	-60%	17%	32%
0.6	120%	-20%	14%	24%
0.4	80%	20%	10%	16%
0.2	40%	60%	7%	8%
0.0	0%	100%	4%	0%

Table 1. Optimal security allocations for different risk tolerances.

3 Capital Market Theory

Portfolio theory is normative in the sense that it produces various decision rules, or heuristics, concerning how a risk averse investor can select portfolios which maximize her expected utility. Next, we consider the implications of this kind of portfolio behavior for the pricing of risk in the capital markets.

Rather than limit one's selection to the efficient set of risky portfolios, we allow investors to combine investment in a mean-variance efficient portfolio along with borrowing or lending at the riskless rate of interest. Thus the portfolio selection problem initially involves determining which risky mean-variance efficient portfolio to select, and it subsequently involves determining the level of exposure (α) which one has to this risky portfolio. Figure 3 illustrates how two different investors with different optimal values for α make such a choice.

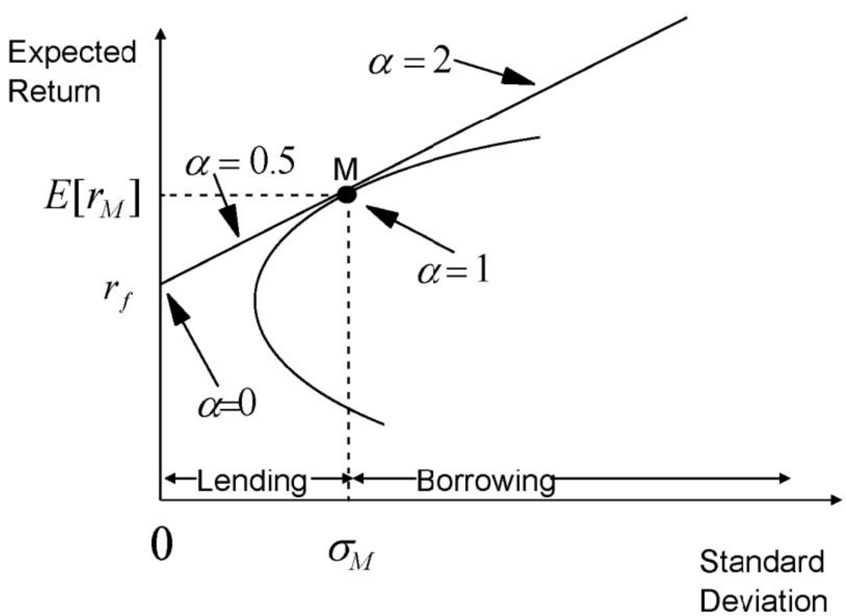

Figure 3. Optimal portfolio selection in equilibrium for two different investors.

From equation (20), investors will determine which of the available mean-variance efficient portfolios maximizes the Sharpe Ratio. Since investors have homogenous beliefs and markets must clear (i.e., supply and demand of securities must be equal), investors will be unanimous in their choice of the optimal risky mean-variance efficient portfolio. The optimal portfolio for all investors will be a hypothetical "market" portfolio which is essentially a value-weighted index fund consisting of all securities in proportion to their market values. This will be the optimal portfolio for all investors because it provides investors with the best possible Sharpe Ratio. Although investors select from the same two securities (the riskless security and the market portfolio), they differentiate themselves according to their preferred level of exposure to the market portfolio (α).

In the case illustrated by Figure 3, one investor solves equation (20) using the parameters for the riskless security and the market portfolio, and finds that her optimal α is 50%. This means that she will select a "lending" portfolio in which 1-α = 50% of her money is invested in the riskless security, and α = 50% is invested in the market portfolio. The other investor finds that her optimal α is 200%. This means that she will select a "borrowing" portfolio in which she doubles her money in the market portfolio by establishing a margin account equal to her net equity investment; i.e., 200% of her equity is invested, and half of the financing comes from short selling the riskless security.

The line in Figure 3 is commonly referred to as the Capital Market Line. This is the efficient frontier in a world in which investors can borrow and lend money at the riskless rate of interest. The equation of the Capital Market Line is:

$$E(r_p) = r_f + \frac{E(r_j) - r_f}{\sigma_j}\sigma_M. \tag{21}$$

The expected rate of return on a mean-variance efficient portfolio therefore comprises two separate components: 1) the return on a riskless security, like U.S. Treasury bills that compensates investors for the time value of money, and 2) a risk premium which compensates investors for bearing risk.

3.1 All risk-return tradeoffs are equal

An important implication of the Capital Market Line is that in equilibrium, all risk-return tradeoffs must be equal. Assume that the market portfolio consists of all (N) securities in the economy, and security j accounts for w_j percent of the market portfolio. Then the equations for the expected return ($E(r_M)$) and the variance ($Var(r_M)$) are given by equations (22) and (23):

$$E(r_M) = \sum_{j=1}^{N} w_j(E(r_j) - r_f) + r_f, \text{and} \tag{22}$$

$$Var(r_M) = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j Cov(r_i, r_j). \tag{23}$$

Suppose we marginally increase w_j in our portfolio. This will cause expected return to change by $E(r_j) - r_f$, and variance to change by $Cov(r_j, r_M)$.[7] Thus, the return/risk trade-off related

[7]From equation (22), note that the impact of a "marginal" increase in w_j on $E(r_M)$ is calculated by differentiating $E(r_M)$ with respect to w_j; thus, $\partial E(r_M)/\partial w_j = E(r_j) - r_f$. From equation (23), the impact

to a marginal increase in w_j corresponds to $(E(r_j)-r_f)/Cov(r_j, r_M)$. In equilibrium, this risk-return tradeoff must be the same for all securities; i.e., $(E(r_i)-r_f)/Cov(r_i, r_M) = (E(r_j)-r_f)/Cov(r_j, r_M)$ for all i and j. If $(E(r_i)-r_f)/Cov(r_i, r_M) \neq (E(r_j)-r_f)/Cov(r_j, r_M)$, then there is an arbitrage opportunity. Suppose $(E(r_i)-r_f)/Cov(r_i, r_M) > (E(r_j)-r_f)/Cov(r_j, r_M)$. Then security i offers a better risk-return tradeoff than security j. Investors will recognize this and respond by purchasing security i and selling security j. Consequently, in equilibrium, the risk-return tradeoff must be equal for all securities; i.e., $(E(r_i)-r_f)/Cov(r_i, r_M) = (E(r_j)-r_f)/Cov(r_j, r_M)$ for all i and j.

3.2 The Capital Asset Pricing Model (CAPM)

If the risk-return tradeoff is the same for all i and j, then the risk-return tradeoff for the market portfolio must also be the same as for i and j; i.e.,

$$\frac{E(r_i)-r_f}{Cov(r_i, r_M)} = \frac{E(r_j)-r_f}{Cov(r_j, r_M)} = \frac{E(r_M)-r_f}{Var(r_M)} \tag{24}$$

Next, we solve equation (24) for the expected returns on securities i and j, relative to the expected return on the market:

$$E(r_i) = r_f + \frac{Cov(r_i, r_M)}{Var(r_M)}(E(r_M)-r_f) = r_f + \beta_i(E(r_M)-r_f), \text{and} \tag{25}$$

$$E(r_j) = r_f + \frac{Cov(r_j, r_M)}{Var(r_M)}(E(r_M)-r_f) = r_f + \beta_j(E(r_M)-r_f), \tag{26}$$

where $\beta_i = \dfrac{Cov\,(r_i, r_M)}{Var\,(r_M)}$ and $\beta_j = \dfrac{Cov\,(r_j, r_M)}{Var\,(r_M)}$. Equations (25) and (26) represent Capital Asset Pricing Model (CAPM) equations for securities i and j; Figure 4 depicts the security market line (SML), which graphically represents the CAPM:

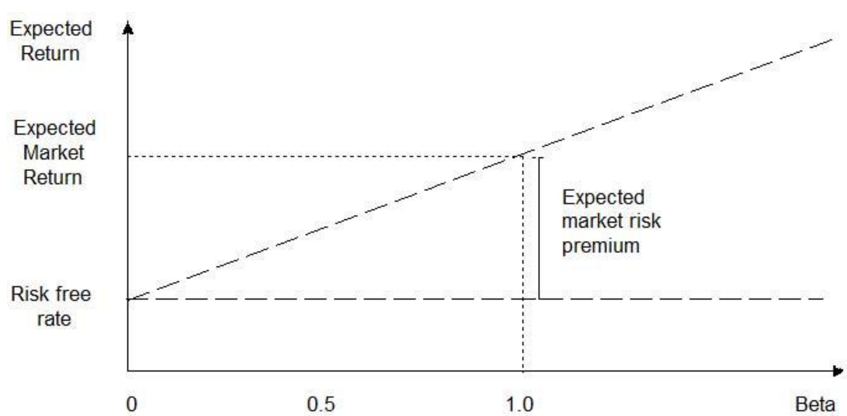

Figure 4. Capital Asset Pricing Model (CAPM)

of a "marginal" increase in w_j on $Var(r_M)$ is calculated by differentiating $Var(r_M)$) with respect to w_j; thus, $\partial Var(r_M)/\partial w_j = \sum_{i=1}^{N} w_i Cov(r_j, r_i) = Cov(r_j, r_M)$.

According to equations (25) and (26), the equilibrium expected rates of return on securities i and j consist of two separate components: 1) the return on a riskless security, and 2) a risk premium which is proportional to a standardized measure of each security's covariance or "systematic" risk, which is measured by its beta. As noted earlier, this theory fits well with the Law of Large Numbers; i.e., an individual security's contribution to portfolio risk is its covariance risk, not its individual variance risk. Therefore, risk averse investors only require compensation for covariance risk.

Suppose that for security i:

$$E(r_i) < r_f + \beta_i(E(r_M) - r_f). \tag{27}$$

Consider the following definition of expected return:

$$E(r_i) = \frac{E(P_i^1 + D_i^1) - P_i^0}{P_i^0}, \tag{28}$$

where P_i^1 is next period's expected price for security i, D_i^1 is security i*'s* expected dividend, and P_i^0 is its current price. Since, as indicated by equation (27), $E(r_i)$ is too low given the level of risk, there will be excess supply of security i. Thus P_i^0 will be bid down and $E(r_i)$ will increase until markets clear and the security market line equation (equation 25) holds. On the other hand, if $E(r_i) > r_f + \beta_i(E(r_M) - r_f)$, then this implies that $E(r_i)$ is too high given the level of risk, which in turn implies that security i is undervalued. Thus P_i^0 will be bid up and $E(r_i)$ will decrease until markets clear and the security market line equation holds once again. Figure 5 provides a graphical illustration of this equilibrium process for two (initially mispriced) securities, each of which have the same beta:

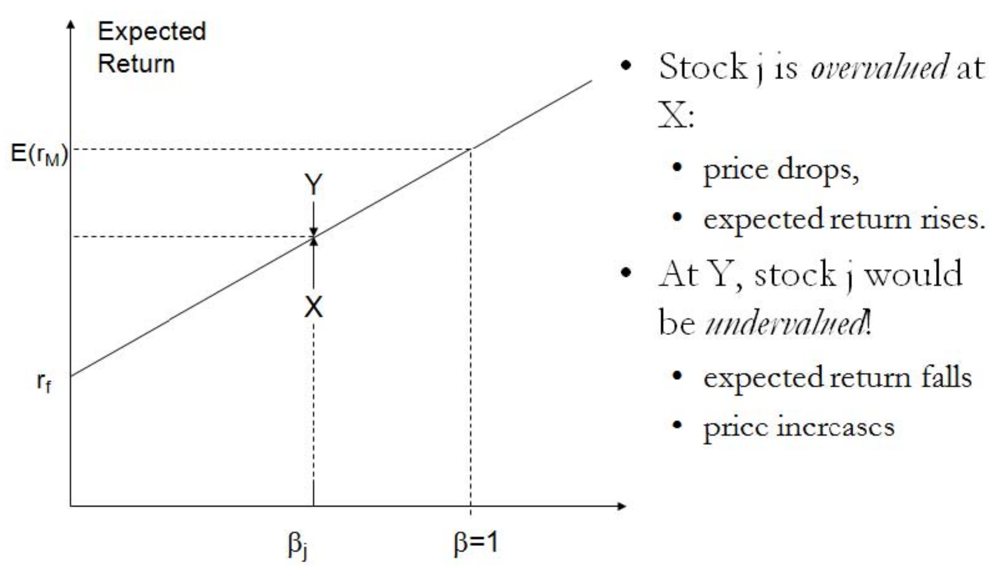

Figure 5. The equilibrium process.

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Appendix

Expected Utility, Mean-Variance Theory, and
Arrow-Pratt Risk Aversion Measures

Expected Utility & Mean-Variance Theory

The logical connection between expected utility and the mean-variance model is made apparent by considering a second-order Taylor series of an arbitrary risk averse utility function $U(W)$ in which $U(W)$ is approximated for values of W which deviate from the expected value of wealth $E(W)$:

$$U(W) \cong U(E(W)) + U'(W - E(W)) + (1/2)U''(W - E(W))^2. \quad \text{(A.1)}$$

Next, we find expected utility by calculating the expected value of equation (A.1):

$$\begin{aligned} E(U(W)) &\cong U(E(W)) + U'E(W - E(W)) + (1/2)U''E(W - E(W))^2 \\ &\cong U(E(W)) + (1/2)U''E(W - E(W))^2 \cong U(E(W)) + (1/2)U''\sigma_W^2. \end{aligned} \quad \text{(A.2)}$$

Since risk aversion implies that marginal utility is positive ($U' > 0$) and diminishing in wealth ($U'' < 0$), it follows from equation (A.2) that $E(U(W))$ is positively related to $E(W)$ and negatively related to σ_W^2. In other words, the mean-variance model obtains as a *special case* of the expected utility model, where higher order moments for W such as skewness and kurtosis are inconsequential.[8]

Arrow-Pratt Risk Aversion Measures

Next, we turn our attention to the Arrow (1965) and Pratt (1964) risk aversion measures. Within the expected utility framework, the expected utility of wealth is equal to the utility of the certainty-equivalent of wealth; i.e., $E(U(W)) = U(W_{CE}) = U[(E(W) - \lambda(E(W)]$, where W_{CE} corresponds to the certainty-equivalent of wealth and λ corresponds to the risk premium. The utility of the certainty-equivalent of wealth can be linearly approximated by a first order Taylor series expansion centered at the expected value of wealth; i.e.,

$$U[E(W) - \lambda(E(W)] \cong U(E(W)) - \lambda(E(W))U'. \quad \text{(A.3)}$$

Next, we set the expected utility of wealth (as shown by equation (A.2)) equal to the utility of the certainty-equivalent of wealth (as shown by equation (A.3)) and solve for $\lambda(E(W))$:

$$\begin{aligned} U(E(W)) + .5U''\sigma_W^2 &= U(E(W)) - \lambda(E(W))U' \\ \Rightarrow \lambda(E(W)) &= -.5\sigma_W^2 (U''/U')|_{W=E(W)} = .5\sigma_W^2 R_A(E(W)), \end{aligned} \quad \text{(A.4)}$$

where $R_A(W) = -U''/U'$ represents the *Arrow-Pratt measure of absolute risk aversion* and $R_A(E(W))$ corresponds to the evaluation of this measure at the expected value of wealth. Absolute risk aversion determines the *dollar amount* of wealth that an investor is willing

[8] In the case of the normal distribution, this is certainly a valid assumption.

to put at risk, whereas *relative risk aversion* $R_R = WR_A(W)$ determines the *proportion* of wealth that an investor is willing to put at risk.[9]

Risk tolerance (τ) corresponds to the reciprocal of relative risk aversion; i.e., $\tau = 1/R_R$. Since the certainty-equivalent of wealth $W_{CE} = (E(W) - \lambda(E(W))$, it follows that the certainty-equivalent of the percentage change in wealth (r_p^c), is equal to the difference between the expected return on the investor's portfolio ($E(r_p)$) less the risk premium (expressed in percentage terms) for this portfolio ($.5\sigma_p^2 R_R = .5\sigma_p^2/\tau$). Therefore, the certainty-equivalent return for portfolio p is written as:

$$r_p^c = E(r_p) - .5\sigma_p^2/\tau. \tag{A.5}$$

[9]Conveniently, most of the utility functions considered in Finance 4335 are characterized by *decreasing* absolute risk aversion (which implies that investors become less risk averse in dollar terms with increases in wealth) and *constant* relative risk aversion (which implies that investors prefer to put a fixed proportion of wealth at risk).