Statistics Class Problem Solutions

by James R. Garven

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Suppose the return distributions for two risky assets are:

State	p_s	$r_{a,s}$	$r_{b,s}$
1	1/3	-3%	36%
2	1/3	9%	-12%
3	1/3	21%	12%

1. Calculate the expected returns for assets a and b.

$$E(r_a) = \sum_{s=1}^{n} p_s r_{a,s} = (1/3)(-3\%) + (1/3)(9\%) + (1/3)(21\%) = 9\%$$
$$E(r_b) = \sum_{s=1}^{n} p_s r_{b,s} = (1/3)(36\%) + (1/3)(-12\%) + (1/3)(12\%) = 12\%$$

2. Calculate the variances and standard deviations for assets a and b.

$$\sigma_a^2 = \sum_{s=1}^{n} p_s (r_{a,s} - E(r_a))^2$$
$$= (1/3)(-3\% - 9\%)^2 + (1/3)(9\% - 9\%)^2 + (1/3)(21\% - 9\%)^2 = .96\%$$
$$\sigma_a = \sqrt{.96\%} = 9.8\%$$
$$\sigma_b^2 = \sum_{s=1}^{n} p_s (r_{a,s} - E(r_a))^2$$
$$= (1/3)(36\% - 12\%)^2 + (1/3)(-12\% - 12\%)^2 + (1/3)(12\% - 12\%)^2 = 3.84\%$$
$$\sigma_b = \sqrt{3.84\%} = 19.6\%$$

3. Calculate the covariance and correlation between assets a and b.

$$\sigma_{ab} = \sum_{s=1}^{n} p_s (r_{a,s} - E(r_a))(r_{b,s} - E(r_b))$$
$$= (1/3)(-12\%)(24\%) + (1/3)(0\%)(-24\%) + (1/3)(12\%)(0) = -.96\%$$
$$\rho_{ab} = \frac{\sigma_{ab}}{\sigma_a \sigma_b} = \frac{.96\%}{(9.8\%)(19.6\%)} = -.50$$

4. Calculate the expected return and standard deviation for an equally weighted portfolio comprising asset a and b.

$$E(r_p) = \sum_{s=1}^{n} w_i E(r_i) = .5(9\%) + .5(12\%) = 10.5\%$$
$$\sigma_p = \sqrt{w_a^2 \sigma_a^2 + w_b^2 \sigma_b^2 + 2 w_a w_b \sigma_{ab}}$$
$$= \sqrt{.25(.96\%) + .25(3.84\%) + 2(.5)(.5)(-.96\%)} = 8.49\%$$

5. Determine the least risky combination of assets a and b and calculate the expected return and standard deviation for such a portfolio.

We can rewrite $\sigma_p^2 = w_a^2\sigma_a^2 + w_b^2\sigma_b^2 + 2w_a w_b \sigma_{ab} = w_a^2\sigma_a^2 + (1-w_a)^2\sigma_b^2 + 2w_a(1-w_a)\sigma_{ab}$. Our task is to select w_a such that σ_p^2 is minimized. We do this by differentiating σ_p^2 with respect to w_a, setting the resulting equation to 0, and solving for w_a. Thus, the following expression obtains:

$$w_a = \frac{\sigma_b^2 - \sigma_{ab}}{\sigma_a^2 + \sigma_b^2 - 2\sigma_{ab}} = \frac{3.84\% + .96\%}{.96\% + 3.84\% + 2(.96\%)} = 5/7;\ w_b = 1 - w_a = 2/7$$

$$E(r_p) = \sum_{s=1}^{n} w_i E(r_i) = (5/7)(9\%) + (2/7)(12\%) = 9.86\%$$

$$\sigma_p = \sqrt{w_a^2\sigma_a^2 + w_b^2\sigma_b^2 + 2w_a w_b \sigma_{ab}}$$

$$= \sqrt{(25/49)(.96\%) + (4/49)(3.84\%) + 2(5/7)(2/7)(-.96\%)} = 6.41\%$$

Note that the coefficient of variation, given by the ratio $\sigma_p/E(r_p)$, is .81 for the equally weighted portfolio compared with .65. This means that to earn an additional 64 basis points in extra return, the investor must take on proportionately greater risk. Whether an investor will take on the extra risk to earn the extra return depends upon how risk averse the investor is (more on that later!).